Approximating Total Weighted Completion Time on Identical Parallel Machines with Precedence Constraints and Release Dates

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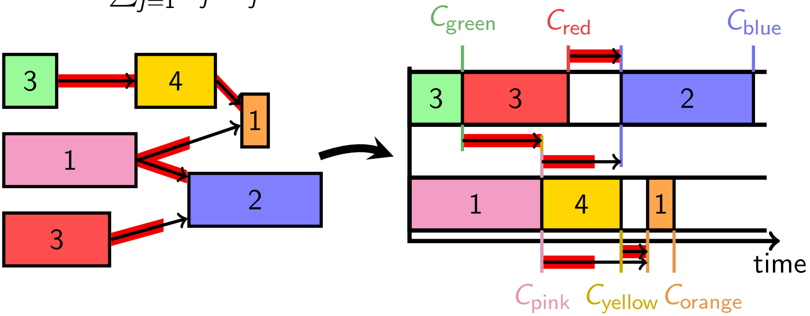
Parallel Machine Scheduling with Precedence Delays

Given: Set $N = \{1, \ldots, n\}$ of jobs with processing times $p_j \geq 0$ and weights $w_j \geq 0$; strict partial order $\prec$ on N with precedence delays $d_{jk} \geq 0$ for every $j \prec k$; number m of machines.

Task: Process each job j non-preemptively for p_j time units on one machine such that, whenever $j \prec k$, job k is started at least d_{jk} time units after j's completion, and the total weighted completion time $\sum_{j=1}^{n} w_j \cdot C_j$ is minimized.



$$3 \cdot C_{green} + C_{pink} + 3 \cdot C_{red} + 4 \cdot C_{yellow} + 2 \cdot C_{blue} + C_{orange}$$

Special Cases

Strongly NP-hard special cases

- no precedence constraints ($P||\sum w_j C_j$)
- unweighted precedence-constrained single-machine scheduling ($1|\text{prec}|\sum C_j$)
  - Set $d_{jk} = 0$ for all $j \prec k$.
  - no 4/3-approximation [Hoogeveen, Schuurman, Woeginger '01]
- unweighted single-machine scheduling with release dates ($1|r_j|\sum C_j$)
  - Add dummy job 0 with $p_0 = 0$ and precedence constraints $0 \prec j$ with $d_{0j} = r_j$ for all $j \in N$.
- parallel machine scheduling with makespan objective ($P||C_{\max}$)
  - Add dummy job $n+1$ with $p_{n+1} = 0$ and $w_{n+1} = 1$ and constraints $j \prec n+1$ for all $j \in N$.

History

Best known approximation algorithms

	prec	r_j, prec	prec. delays	Reference
1996		13		Hall, Shmoys, Wein
		7		Hall, Schulz, Shmoys, Wein
		5.328		Chakrabarti et al.
1998			4	Munier, Queyranne, Schulz
2017	3.386			Li
2018			3.386	

Time-Indexed LP-Relaxation

Assumption: $p_j \in \mathbb{N}, d_{jk} \in \mathbb{Z}_{\geq 0}$, $T \in \mathbb{N}$ upper bound on makespan.

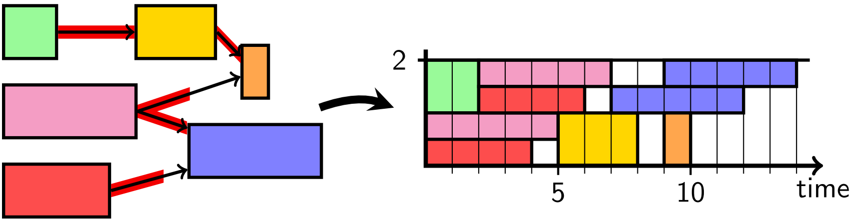
Variables

- $x_{jt} \geq 0$ indicating whether job j completes at time t
- $C_j \geq 0$ indicating completion time of job j

Constraints

- Relation between variables: $$C_j = \sum_{t=1}^{T} t \cdot x_{jt} \quad \forall j \in N$$
- Each job is scheduled: $$\sum_{t=1}^{T} x_{jt} = 1 \quad \forall j \in N$$
- Machine capacity: $$\sum_{j=1}^{n} \sum_{t=s}^{s+p_j-1} x_{jt} \leq m \quad \forall s \in \{1, \ldots, T\}$$
- Precedence delays: $$\sum_{t=1}^{s+d_{jk}+p_k} x_{kt} \leq \sum_{t=1}^{s} x_{jt} \quad \forall j \prec k,\ s \in \{0, \ldots, T\}$$
- Time starts at 0: $$x_{jt} = 0 \quad \forall j \in N,\ t \in \{1, \ldots, p_j - 1\}$$

LP solutions



$C_{\text{green}} = 2$

$C_{\text{pink}} = \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 7 = 6$

$C_{\text{red}} = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 6 = 5$

$C_{\text{yellow}} = 8$

$C_{\text{orange}} = 10$

$C_{\text{blue}} = \frac{1}{2} \cdot 12 + \frac{1}{2} \cdot 14 = 13$

List Scheduling in Order of α-Points

Let $(x^{\mathsf{LP}}, C^{\mathsf{LP}})$ be an optimal solution to the LP relaxation.

Definition

For $j \in N$ and $\alpha \in [0,1]$ the LP-α-point of j is

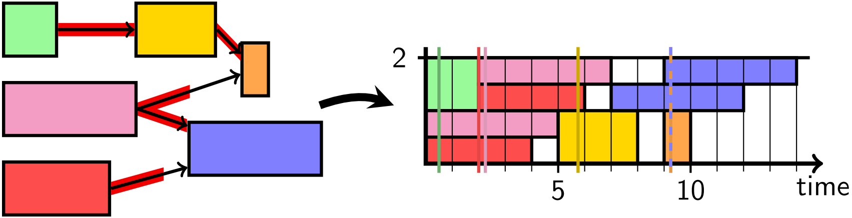
$$C_j^{\mathsf{LP}}(\alpha) \coloneqq C_j^{\mathsf{LP}} - (1-\alpha) \cdot p_j.$$

Job-driven list scheduling in order of LP-α-points

In non-increasing order of $C_j^{\mathsf{LP}}(\alpha)$ process each job as early as possible s.t.

- all precedence delays are obeyed.
- j is scheduled after all jobs already assigned to the same machine.

Example ($\alpha = 1/4$)



$C^{LP}_{green}(\tfrac{1}{4}) = 2 - \tfrac{3}{4} \cdot 2 = 0.5$

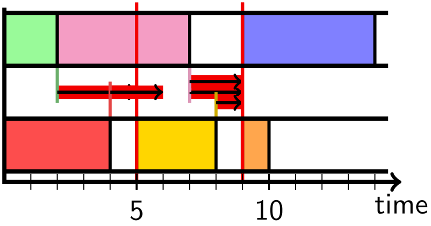
$C^{LP}_{pink}(\tfrac{1}{4}) = 6 - \tfrac{3}{4} \cdot 5 = 2.25$

$C^{LP}_{red}(\tfrac{1}{4}) = 5 - \tfrac{3}{4} \cdot 4 = 2$

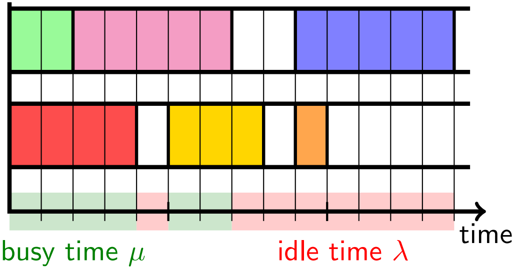
$C^{LP}_{yellow}(\tfrac{1}{4}) = 8 - \tfrac{3}{4} \cdot 3 = 5.75$

$C^{LP}_{orange}(\tfrac{1}{4}) = 10 - \tfrac{3}{4} \cdot 1 = 9.25$

$C^{LP}_{blue}(\tfrac{1}{4}) = 13 - \tfrac{3}{4} \cdot 5 = 9.25$



Bounding the Completion Times



Consider partial schedule when the blue job has been assigned.

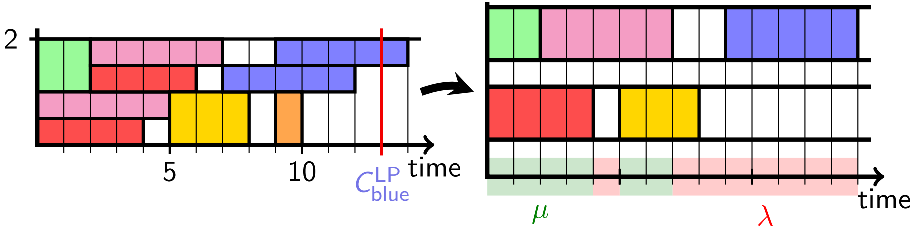
$$C_{\text{blue}} = \mu + \lambda.$$

For $\alpha \in (0, \frac{1}{2}]$ it holds that

$$\mu \leq \frac{1}{\alpha} \cdot C_{\text{blue}}^{\text{LP}}, \qquad \lambda \leq \frac{1}{1-\alpha} \cdot C_{\text{blue}}^{\text{LP}}.$$

For $\alpha = \frac{1}{2}$ this gives a 4-approximation algorithm.

Bounding the Busy Time



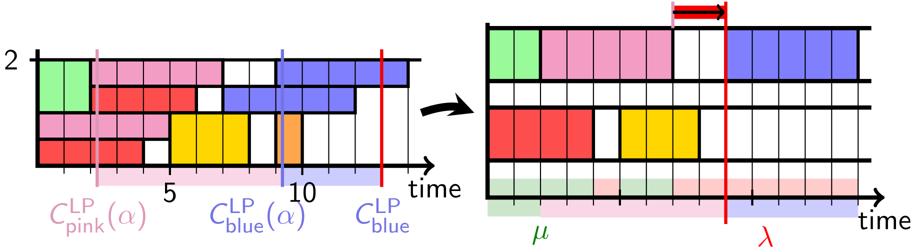
Let $\alpha \in (0, \frac{1}{2}]$ and $J_\alpha := \{j \in N \mid C_j^{\text{LP}}(\alpha) \leq C_{\text{blue}}^{\text{LP}}\}$.

Upper bound on busy time: $\mu \leq \dfrac{p(J_\alpha)}{m}$.

Lower bound on LP-completion time: $C_{\text{blue}}^{\text{LP}} \geq \alpha \cdot \dfrac{p(J_\alpha)}{m}$.

$$\mu \leq \frac{1}{\alpha} \cdot C_{\text{blue}}^{\text{LP}}.$$

Bounding the Idle Time



Trace back why the blue job is not completed earlier.

- The blue job has to be processed (can be idle time).
- The blue job has to wait for the pink job (can be idle time).
- The pink job has to wait because before all machines are busy (cannot be idle time).

$$\lambda \leq \underbrace{p_{\mathrm{blue}}}_{=\frac{1}{1-\alpha}(C^{\mathrm{LP}}_{\mathrm{blue}}-C^{\mathrm{LP}}_{\mathrm{blue}}(\alpha))} + \underbrace{d_{\mathrm{pink,blue}} + p_{\mathrm{pink}}}_{\leq\frac{1}{1-\alpha}(C^{\mathrm{LP}}_{\mathrm{blue}}(\alpha)-C^{\mathrm{LP}}_{\mathrm{pink}}(\alpha))} \leq \frac{1}{1-\alpha}\cdot C^{\mathrm{LP}}_{\mathrm{blue}}.$$

List Scheduling in Order of Random α-Points

$$C_{\text{blue}} = \mu + \lambda.$$

For $\alpha \in (0, \frac{1}{2}]$ it holds that

$$\mu \leq \frac{1}{\alpha} \cdot C^{\text{LP}}_{\text{blue}}, \qquad \lambda \leq \frac{1}{1-\alpha} \cdot C^{\text{LP}}_{\text{blue}}.$$

Observations

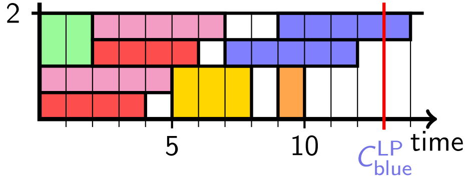
- For small α the idle time bound gets better, but the busy time bound gets worse.
- The busy time bound cannot be tight for all $\alpha \in (0, \frac{1}{2}]$ simultaneously.

Idea: Choose $\boldsymbol{\alpha}$ uniformly at random from $(0, \frac{1}{2}]$. Then

$$\mathrm{E}[\boldsymbol{\mu}] \leq 2 \cdot C^{\text{LP}}_{\text{blue}}, \qquad \mathrm{E}[\boldsymbol{\lambda}] \leq 2 \ln 2 \cdot C^{\text{LP}}_{\text{blue}}.$$

Bounding the Expected Busy Time

Overview



For $\alpha \in (0, \frac{1}{2}]$ let $J_\alpha := \{j \in N \mid C_j^{\text{LP}}(\alpha) \le C_{\text{blue}}^{\text{LP}}\}$.

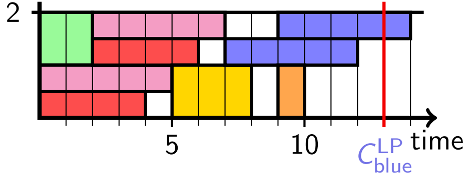
Upper bound on busy time: $\boldsymbol{\mu} \le \dfrac{p(J_{\boldsymbol{\alpha}})}{m}$.

Lower bound on LP-completion time: $C_{\text{blue}}^{\text{LP}} \ge \dfrac{\mathrm{E}[p(J_{\boldsymbol{\alpha}})]}{2m}$.

$$\mathrm{E}[\boldsymbol{\mu}] \le 2 \cdot C_{\text{blue}}^{\text{LP}}.$$

Bounding the Expected Busy Time

Lower bound on LP-completion time



For $\alpha \in (0, \frac{1}{2}]$ let $J_\alpha := \{j \in N \mid C_j^{\text{LP}}(\alpha) \leq C^{\text{LP}}_{\text{blue}}\}$.

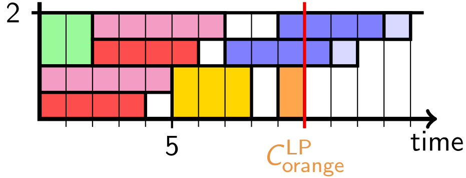
Claim: $C^{\text{LP}}_{\text{blue}} \geq \frac{1}{2} \cdot \frac{\text{E}[p(J_{\boldsymbol{\alpha}})]}{m}$.

In the example $C_j^{\text{LP}}(\frac{1}{2}) \leq C^{\text{LP}}_{\text{blue}}$ for all $j \in N$, whence $J_\alpha = N$ for all α.
Therefore,

$$C^{\text{LP}}_{\text{blue}} \geq \frac{1}{2} \cdot \frac{p(N)}{m} = \frac{1}{2} \cdot \frac{\text{E}[p(J_{\boldsymbol{\alpha}})]}{m}.$$

Bounding the Expected Busy Time

Lower bound on LP-completion time



For $\alpha \in (0, \frac{1}{2}]$ let $J_\alpha := \{j \in N \mid C_j^{\mathrm{LP}}(\alpha) \leq C^{\mathrm{LP}}_{\mathrm{orange}}\}$.

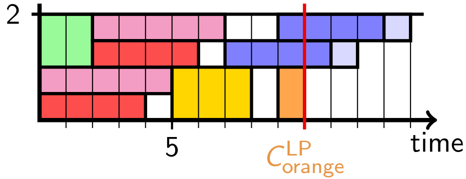
Claim: $C^{\mathrm{LP}}_{\mathrm{orange}} \geq \frac{1}{2} \cdot \frac{\mathrm{E}[p(J_{\boldsymbol{\alpha}})]}{m}$.

$$p(J_{\boldsymbol{\alpha}}) = \begin{cases} p_{\mathrm{green}} + p_{\mathrm{pink}} + p_{\mathrm{red}} + p_{\mathrm{yellow}} + p_{\mathrm{orange}} + p_{\mathrm{blue}} & \text{if } \alpha \leq \frac{2}{5}; \\ p_{\mathrm{green}} + p_{\mathrm{pink}} + p_{\mathrm{red}} + p_{\mathrm{yellow}} + p_{\mathrm{orange}} & \text{if } \alpha > \frac{2}{5}. \end{cases}$$

$$\mathrm{E}[p(J_{\boldsymbol{\alpha}})] = p_{\mathrm{green}} + p_{\mathrm{pink}} + p_{\mathrm{red}} + p_{\mathrm{yellow}} + p_{\mathrm{orange}} + \frac{4}{5} \cdot p_{\mathrm{blue}}$$

Bounding the Expected Busy Time

Lower bound on LP-completion time



For $\alpha \in (0, \frac{1}{2}]$ let $J_\alpha := \{j \in N \mid C_j^{\text{LP}}(\alpha) \leq C^{\text{LP}}_{\text{orange}}\}$.

Claim: $C^{\text{LP}}_{\text{orange}} \geq \frac{1}{2} \cdot \frac{\text{E}[p(J_{\boldsymbol{\alpha}})]}{m}$.

$$\text{E}[p(J_{\boldsymbol{\alpha}})] = p_{\text{green}} + p_{\text{pink}} + p_{\text{red}} + p_{\text{yellow}} + p_{\text{orange}} + \frac{4}{5} \cdot p_{\text{blue}}$$

The modified jobs J' have midpoints bounded by $C^{\text{LP}}_{\text{orange}}$. Therefore,

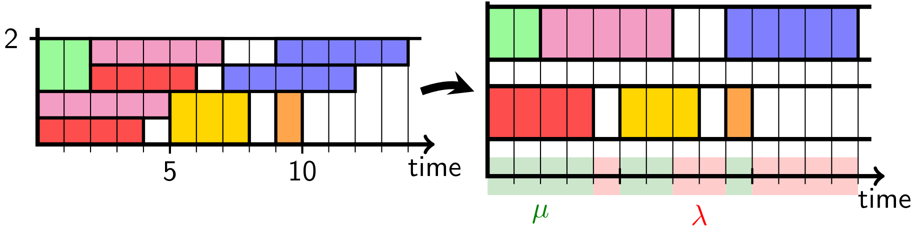
$$C^{\text{LP}}_{\text{orange}} \geq \frac{1}{2} \cdot \frac{p(J')}{m} = \frac{1}{2} \cdot \frac{\text{E}[p(J_{\boldsymbol{\alpha}})]}{m}.$$

Concluding Remarks

- The LP-relaxation has exponential size. One can reduce to polynomial size with a loss of $1+\varepsilon$ by considering exponentially increasing time intervals.
- The algorithm can be derandomized as there is only a polynomial number of combinatorially different values of α.

Open problem

Improve the analysis/algorithm/lower bound.



Thank you.

References

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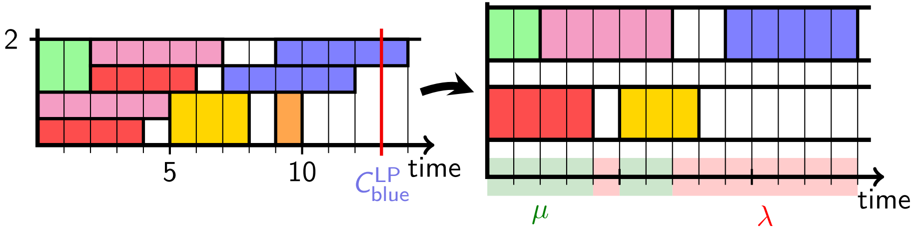
Additional Slides

1. Lower bound on LP-completion time
2. Tight example for fixed α

Bounding the Busy Time



Let $\alpha \in (0, \frac{1}{2}]$ and $J_\alpha := \{j \in N \mid C_j^{\mathrm{LP}}(\alpha) \le C^{\mathrm{LP}}_{\mathrm{blue}}\}$.

Upper bound on busy time: $\mu \le \dfrac{p(J_\alpha)}{m}$.

Lower bound on LP-completion time: $C^{\mathrm{LP}}_{\mathrm{blue}} \ge \alpha \cdot \dfrac{p(J_\alpha)}{m}$.

$$\mu \le \frac{1}{\alpha} \cdot C^{\mathrm{LP}}_{\mathrm{blue}}.$$

Bounding the Busy Time

The following inequalities are valid for the time-indexed LP-relaxation:

$$C_j \geq p_j \qquad \forall j \in N$$

$$\sum_{j \in J} p_j C_j \geq \frac{1}{2m} \cdot \left(\sum_{j \in J} p_j \right)^2 + \frac{1}{2} \sum_{j \in J} p_j^2 \qquad \forall J \subseteq N$$

$$C_k \geq C_j + d_{jk} + p_k \qquad \forall j \prec k$$

Bounding the Busy Time

$$C_j^{\mathrm{LP}}(\tfrac{1}{2}) = C_j^{\mathrm{LP}}(\alpha) + (\tfrac{1}{2} - \alpha)p_j \leq C_j^{\mathrm{LP}}(\alpha) + \frac{\frac{1}{2} - \alpha}{\alpha} \cdot C_j^{\mathrm{LP}}(\alpha) = \frac{1}{2 \cdot \alpha} \cdot C_j^{\mathrm{LP}}(\alpha).$$

This implies that

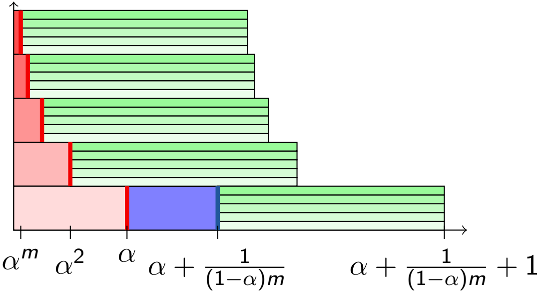
$$\frac{1}{2m} \cdot \left(\sum_{j \in J_\alpha} p_j \right)^2 \leq \sum_{j \in J_\alpha} p_j \cdot C_j^{\mathrm{LP}}(\tfrac{1}{2}) \leq \frac{1}{2 \cdot \alpha} \sum_{j \in J_\alpha} p_j \cdot C_j^{\mathrm{LP}}(\alpha)$$

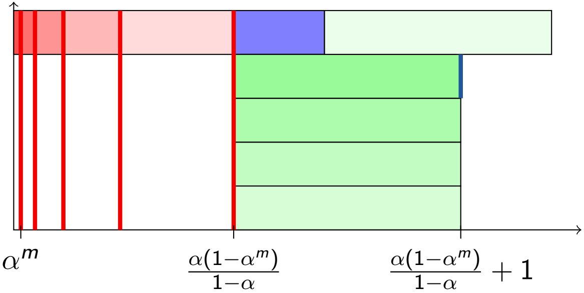
$$\leq \frac{1}{2 \cdot \alpha} \left(\sum_{j \in J_\alpha} p_j \right) \cdot C_{j^*}^{\mathrm{LP}},$$

so that

$$\mu \leq \frac{1}{m} \cdot \sum_{j \in J_\alpha} p_j \leq \frac{1}{\alpha} \cdot C_{j^*}^{\mathrm{LP}}.$$

Tightness for fixed α





$$\frac{\frac{\alpha(1-\alpha^m)}{1-\alpha}+1}{\alpha+\frac{1}{(1-\alpha)m}} \to \frac{1}{1-\alpha} + \frac{1}{\alpha}.$$