Approximation in Deterministic and Stochastic Machine Scheduling

Wissenschaftliche Aussprache

May 06, 2021

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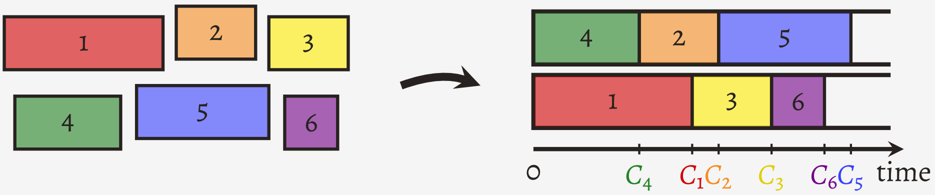
Scheduling on Identical Parallel Machines

Given: jobs $j = 1, \ldots, n$ with processing times $p_j \geq 0$ and weights $w_j \geq 0$, number of machines m;

Task: schedule each job for p_j time units on some machine so as to minimize the sum of weighted completion times $\sum_{j=1}^{n} w_j \cdot C_j$.



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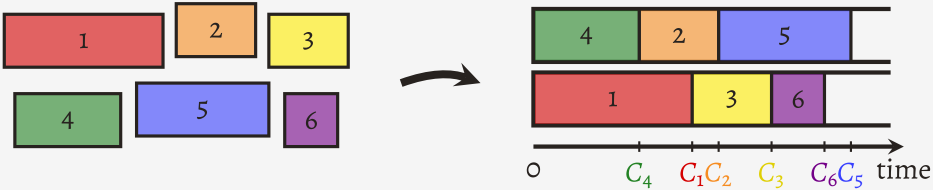

Complexity

- NP-hard for two machines (Bruno et al. 1974)
- strongly NP-hard for arbitrary number of machines (Lageweg, Lenstra 1977)

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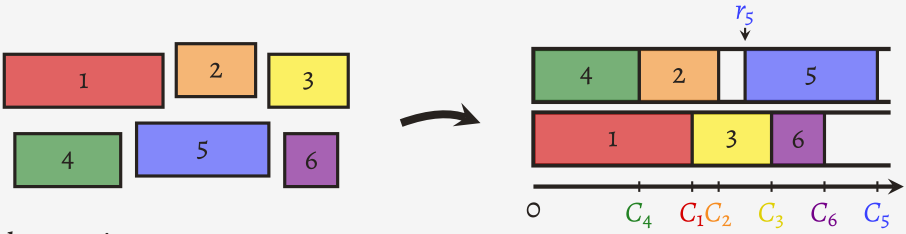

Problem variants

- release dates $r_j \geq 0$

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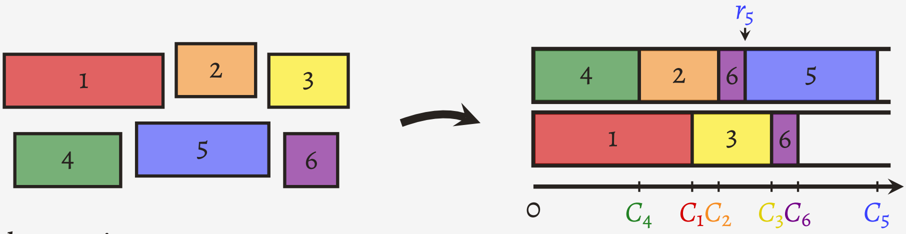

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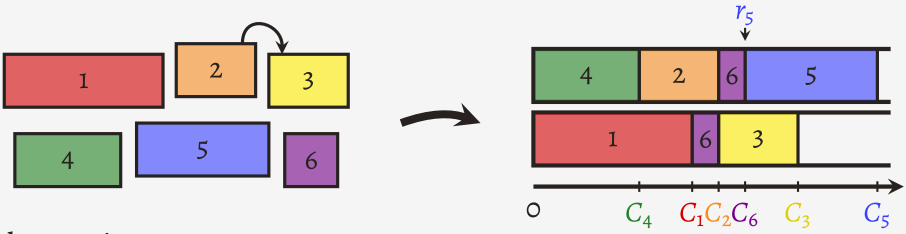

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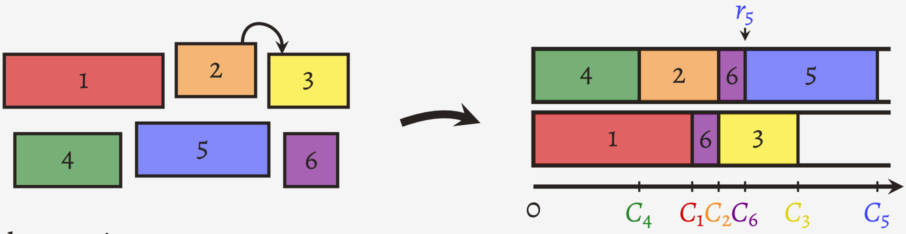

Problem variants

- release dates $r_j \geq 0$
- preemptive vs. non-preemptive
- precedence constraints/delays
- heterogeneous machines

Scheduling under Uncertainty

I. Adversarial models
  - online job arrivals
  - unknown processing times (non-clairvoyant scheduling)

Competitive analysis: compare to offline optimum

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Competitive analysis: compare to offline optimum

II. Stochastic models
- random processing times with known distributions

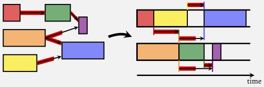
Minimize expected total weighted completion time

Overview

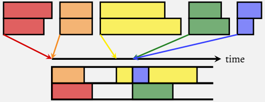
I. Deterministic Scheduling

WSPT Rule

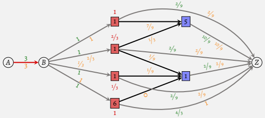
Scheduling with Precedence Delays



Online Non-Migratory Preemptive Scheduling on Unrelated Machines

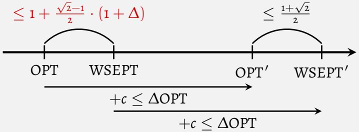


Non-Clairvoyant Online Scheduling with Precedence Delays

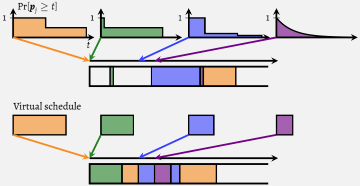


II. Stochastic Scheduling

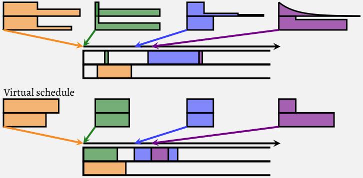
WSEPT Rule



Stochastic Scheduling on Single Machine with Release Dates



Stochastic Online Scheduling on Unrelated Machines



Part I

Deterministic Scheduling

Main Results

WSPT Rule

Scheduling with Precedence Delays

Online Non-Migratory Preemptive Scheduling on Unrelated Machines

Non-Clairvoyant Online Scheduling with Precedence Delays

Weighted Shortest Processing Time First (WSPT) rule

Whenever a machine is free, start available job with maximum ratio w_j/p_j on it.

For a fixed number m of machines, what is the maximum possible approximation ratio of the WSPT rule?

WSPT Rule – Previous Work

Theorem (Kawaguchi, Kyan 1986)

For an arbitrary number of machines WSPT rule has tight approximation ratio $\frac{1+\sqrt{2}}{2} \approx 1.207$.

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- Worst case is attained if all jobs have $w_j = p_j$.
- In this case WSPT is list scheduling according to the input order.
- Worst case is attained when $n, m \to \infty$.

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- Worst case is attained when $n, m \to \infty$.

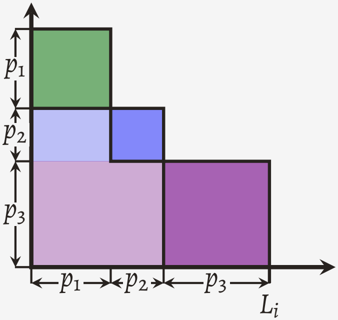
- There is a PTAS for scheduling jobs on an arbitrary number of identical machines. (Skutella, Woeginger 2000)
- For any number m of machines there is an FPTAS for scheduling jobs on m identical machines. (Sahni 1976)

Reduction to Unit Smith Ratio

Lemma (Kawaguchi, Kyan 1986; Schwiegelshohn 2011)

On any number of machines the worst-case approximation ratio is attained when $w_j = p_j$ *for all* j.

Objective Function in Terms of Machine Loads (for $w_j = p_j$)



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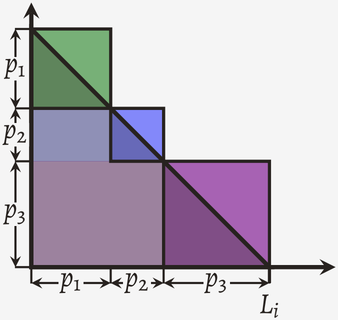

one machine i:

$$\sum_{j \to i} p_j \cdot C_j = \frac{1}{2} \Big(\underbrace{\sum_{j \to i} p_j}_{=:L_i} \Big)^2 + \frac{1}{2} \sum_{j \to i} p_j^2$$

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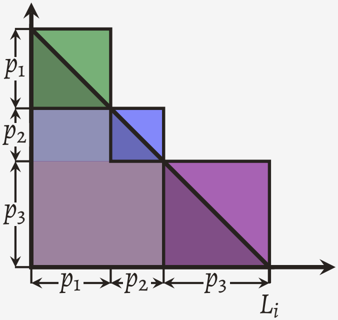

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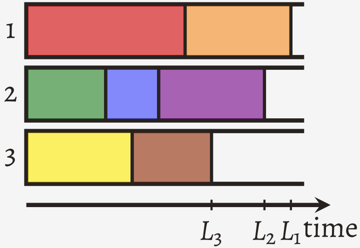
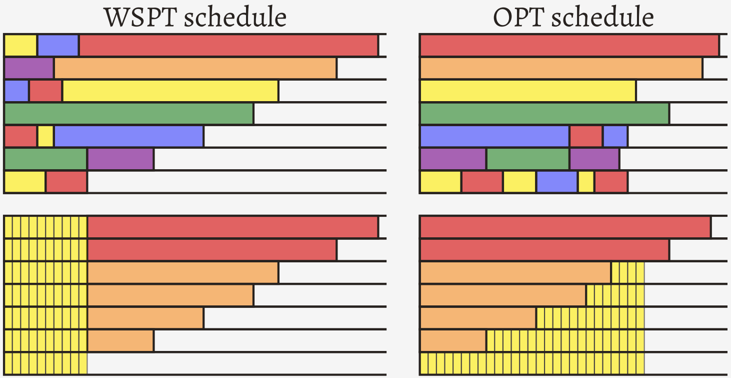


m-machine schedule:

$$\sum_{j=1}^{n} p_j \cdot C_j = \frac{1}{2} \sum_{i=1}^{m} L_i^2 + \frac{1}{2} \sum_{j=1}^{n} p_j^2$$

Reduction to Worst Case



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$\sum_j p_j^2$ decreases by δ

$\sum_i L_i^2$ not increased

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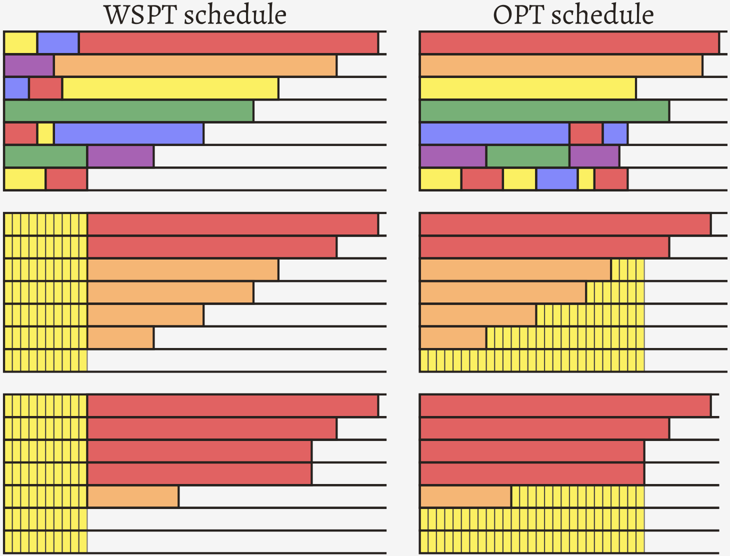

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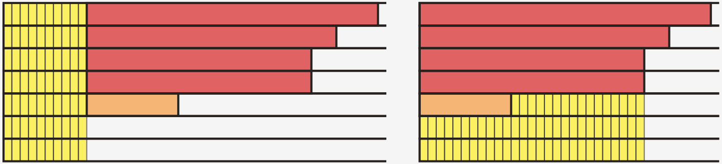

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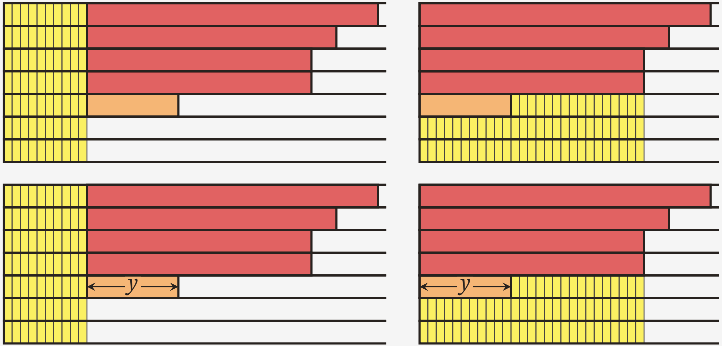

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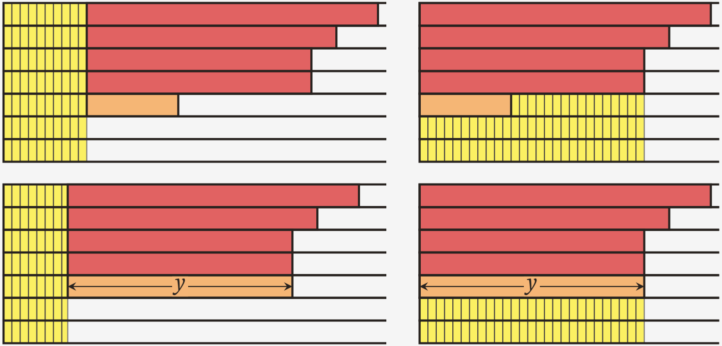

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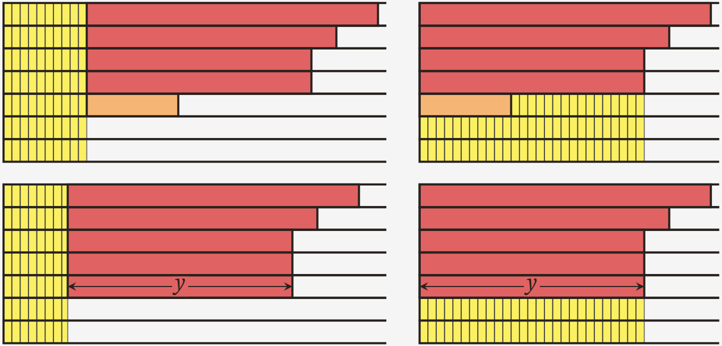

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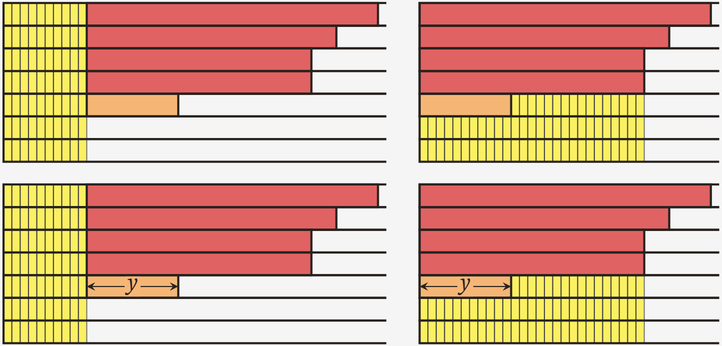

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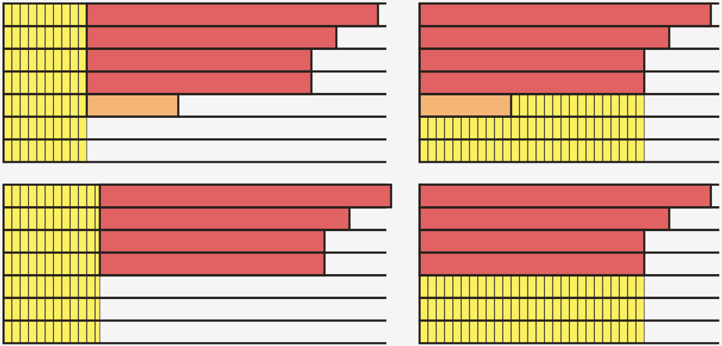

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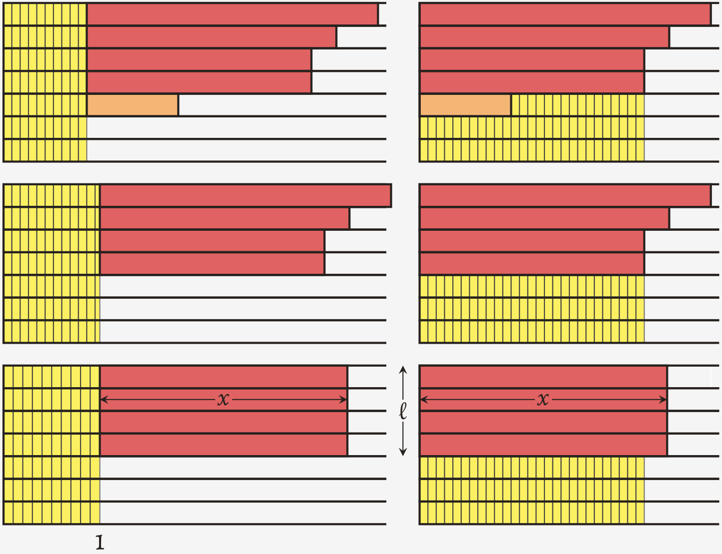

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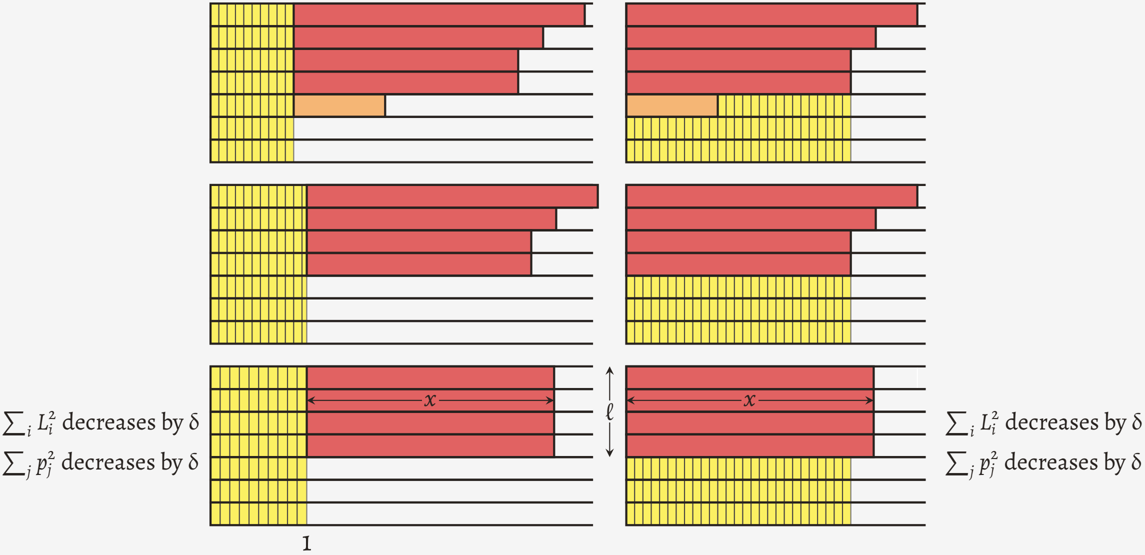


Find worst ℓ, x by calculation.

Approximation Ratio of the WSPT Rule on Fixed Number of Machines

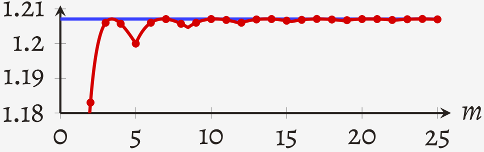
Theorem

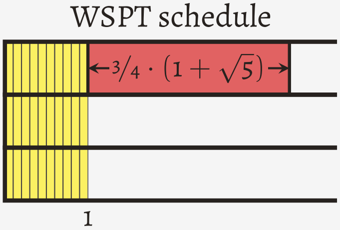
On $m \geq 2$ machines the worst case for the WSPT rule occurs for

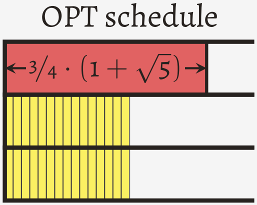
1. m/ε *small jobs with* $w_j = p_j = \varepsilon \quad (\varepsilon \to 0)$,
2. $\ell := \left\lfloor \frac{2m - \sqrt{2m^2-1}}{2} \right\rceil$ *jobs with* $w_j = p_j = \frac{m}{\sqrt{\ell \cdot (2m-\ell)} - \ell}$.

In this case the approximation ratio is

$$1 + \frac{\sqrt{\ell \cdot (2m - \ell)} - \ell}{2m}.$$







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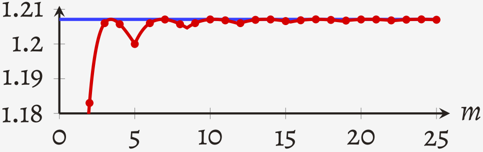

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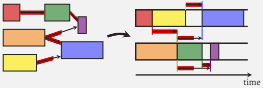

Online Non-Migratory Preemptive Scheduling on Unrelated Machines

Non-Clairvoyant Online Scheduling with Precedence Delays

Theorem (Li 2020)

There is a 3.386-approximation algorithm for scheduling jobs with precedence constraints.

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- includes the setting with precedence constraints and release dates.

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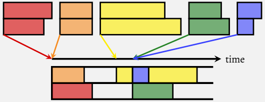

Non-Clairvoyant Online Scheduling with Precedence Delays

Theorem

The Greedy-Assignment WSRPT algorithm is a 4-competitive deterministic online algorithm for non-migratory preemptive scheduling on unrelated machines.

Previous results

- 7.216-competitive deterministic online algorithm (Gupta et al. 2021+)
- 5.771-competitive randomized online algorithm (Hall et al. 1997; Chakrabarti et al. 1996)
- With $1+\varepsilon$ speed augmentation, Greedy-Assignment WSRPT algorithm is $\left(2+\frac{2}{\varepsilon}\right)$-competitive for $\sum w_j(C_j - r_j)$. (Anand et al. 2012)
  $\rightsquigarrow$ 8-competitive alg. for $\sum w_j C_j$. (Bansal, Pruhs 2004)

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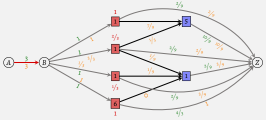


Theorem (Garg et al. 2019)

There is a 10-competitive deterministic non-clairvoyant online algorithm for jobs with precedence constraints.

Theorem

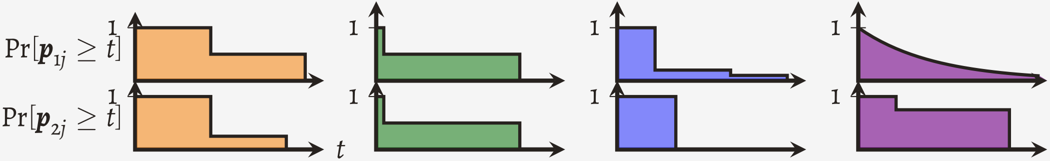
A simplification of the algorithm of Garg et al. is 8-competitive for jobs with precedence delays.

Part II

Stochastic Scheduling

Stochastic Scheduling on Unrelated Parallel Machines

Given: weights $w_j \geq 0$ and distributions of independent random processing times $\boldsymbol{p}_{ij} > 0$ of jobs $j = 1, \dots, n$ on machines $i = 1, \dots, m$



Task: find non-preemptive scheduling policy Π minimizing the expected sum of weighted completion times.

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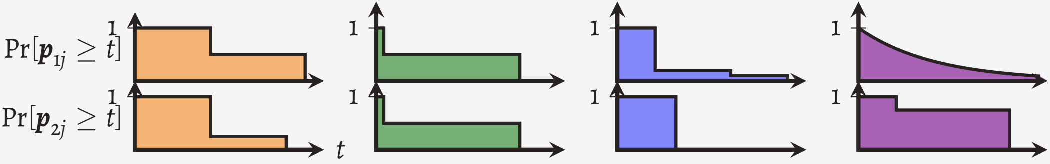

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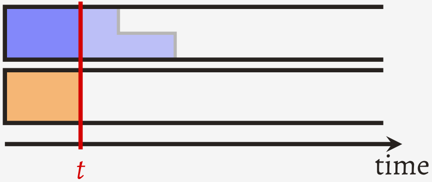

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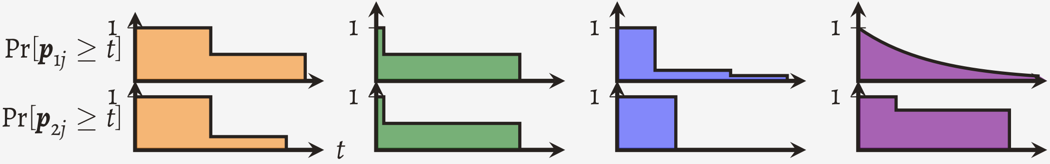

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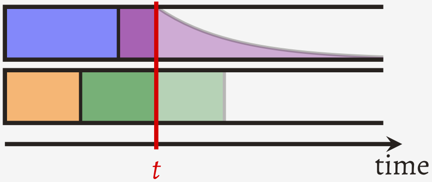

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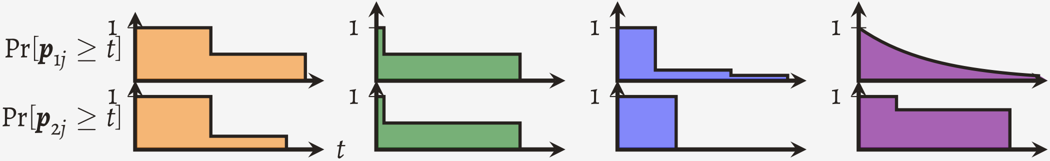

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- function $\mathbb{R}^{mn} \to \mathbb{R};\ (p_{ij}) \mapsto \sum_{j=1}^{n} w_j \cdot C_j^{\Pi}(p_{ij})$ must be measurable.

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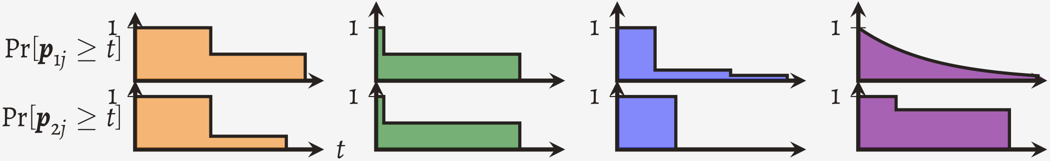

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Alternative definitions of scheduling policy (cp. Möhring, Radermacher 1985)

A) function $\mathbb{R}^{mn} \to \mathbb{R}^n_{\geq 0} \times [m]^n$;

B) function from state space to action space.

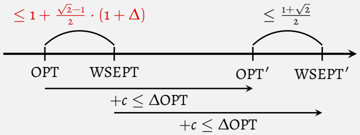
Approximative Policies

- We are interested in simple approximative policies.
- Some performance guarantees depend on an upper bound Δ on the squared coefficients of variation

$$\mathrm{CV}[\boldsymbol{p}_{ij}]^2 = \frac{\mathrm{Var}[\boldsymbol{p}_{ij}]}{\mathrm{E}[\boldsymbol{p}_{ij}]^2}.$$

Main Results

WSEPT Rule



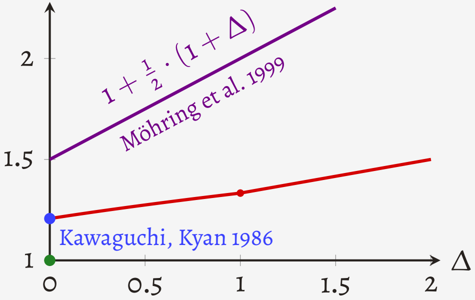
Stochastic Scheduling on Single Machine with Release Dates

Stochastic Online Scheduling on Unrelated Machines

Theorem

The WSEPT rule has performance guarantee

$$\begin{cases} 1 + \frac{1}{2\cdot(1+\sqrt{2(1+\Delta)})} \cdot (1+\Delta) & \text{if } \Delta \leq 1; \\ 1 + \frac{1}{6} \cdot (1+\Delta) & \text{if } \Delta \geq 1. \end{cases}$$



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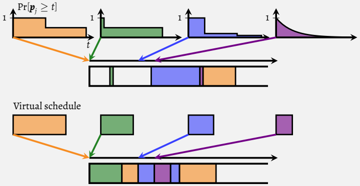
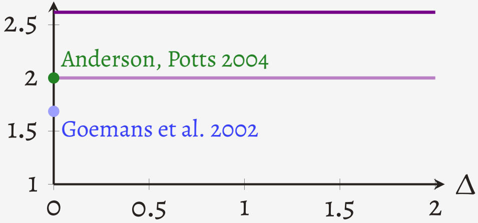


Stochastic Online Scheduling on Unrelated Machines

Theorem (Schulz 2008)

There are a $\left(1 + \max\left\{\phi, \frac{\phi+1}{2}(1+\Delta)\right\}\right)$*-competitive deterministic and a* $(2+\Delta)$*-competitive randomized online scheduling policy for identical machines.*

- on single machine $(\phi + 1)$-competitive (deterministic) and 2-competitive (randomized) $(\phi + 1 \approx 2.618)$



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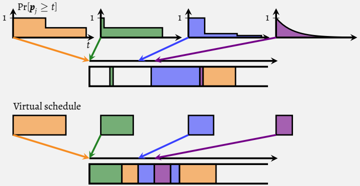

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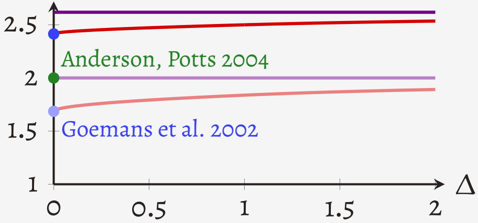

- If Δ is known in advance (*semi-online*), these can be improved.

Main Results

WSEPT Rule

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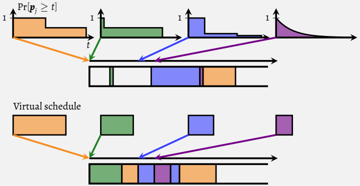
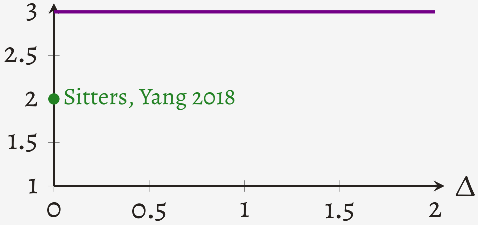


Stochastic Online Scheduling on Unrelated Machines

Theorem (Möhring et al. 1999)

There is an efficient scheduling policy with performance guarantee 3 for scheduling jobs with precedence constraints and release dates on a single machine.



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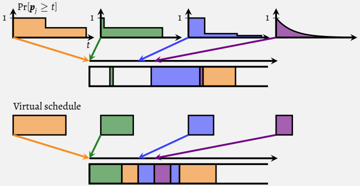

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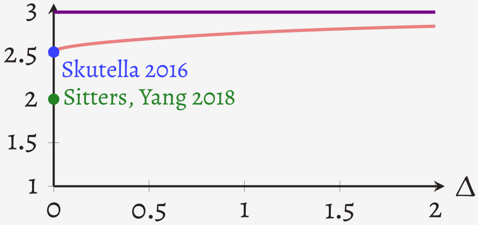

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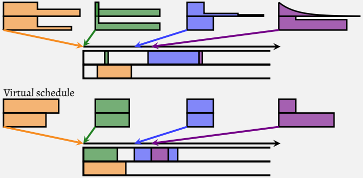


Theorem (Gupta et al. 2021+)

There is a $3.608 \cdot h(\Delta) \cdot (2 + \Delta)$*-competitive deterministic online scheduling policy for unrelated machines, where*

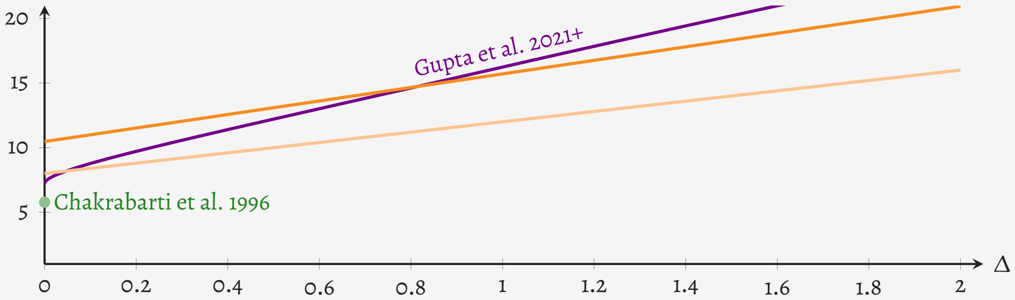
$$h(\Delta) := \begin{cases} 1 + \frac{\sqrt{\Delta}}{2} & \text{if } \Delta \leq 1; \\ 1 + \frac{\Delta}{\Delta+1} & \text{if } \Delta \geq 1. \end{cases}$$

Theorem

There is a $(3 + \sqrt{5}) \cdot (2 + \Delta)$*-competitive deterministic and an* $(8 + 4\Delta)$*-competitive randomized online scheduling policy for unrelated machines.*

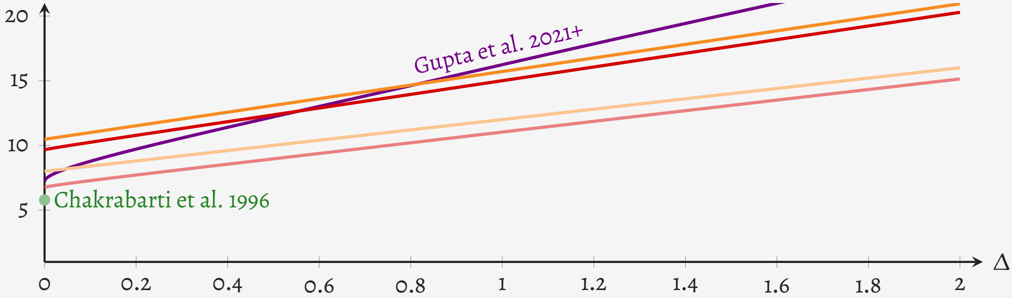
- If Δ is known in advance (semi-online), these can be improved.

Stochastic Online Scheduling on Unrelated Machines



- Jobs j arrive over time;
- at the release date r_j the weight w_j and the distributions of all $\boldsymbol{p}_{ij}$, $i \in [m]$, are given;
- if j is scheduled on machine i, the outcome of $\boldsymbol{p}_{ij}$ becomes known when j completes.
- *Competitive analysis*: compare to optimal scheduling policy.

Stochastic Online Scheduling on Unrelated Machines



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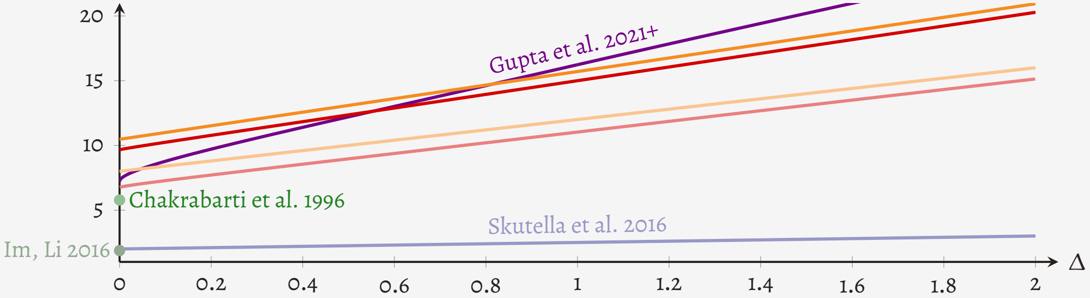

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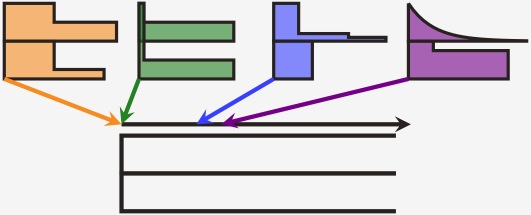
Online Scheduling Policies

- The *mean busy time* M_j of job j is the average of all times when it is being processed.



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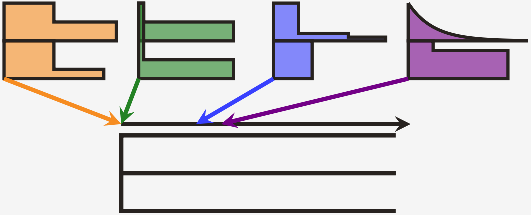
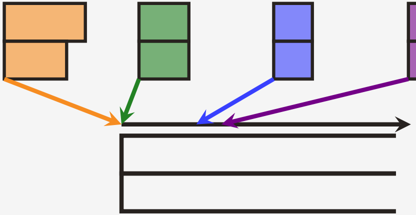
- The *mean busy time* M_j of job j is the average of all times when it is being processed.
- When job j is released, assign it to a machine i with minimum increase of

$$\sum_{j=1}^{n} w_j \cdot \left(M_j + \frac{p_{ij}}{2}\right)$$

in the virtual preemptive WSPT schedule of deterministic counterparts with $p_{ij} := \mathrm{E}[\boldsymbol{p}_{ij}]$.



Virtual schedule



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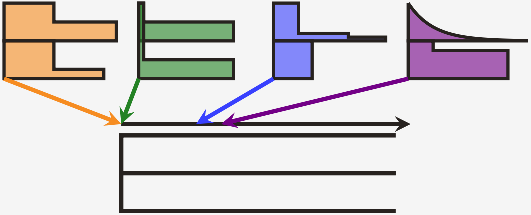

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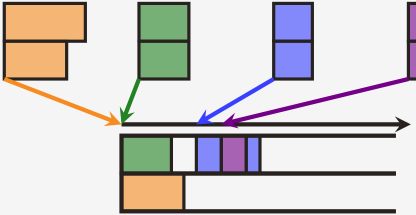

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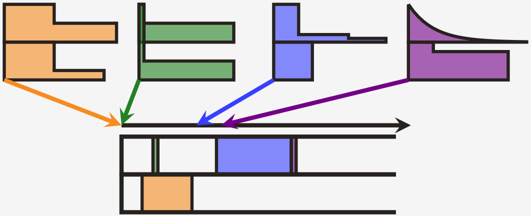
- On each machine, schedule the jobs by single machine (semi-)online policy from previous chapter.



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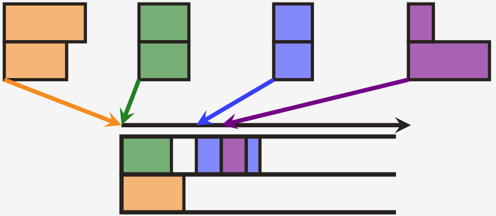


Overview

I. Deterministic Scheduling

WSPT Rule

Scheduling with Precedence Delays

Online Non-Migratory Preemptive Scheduling on Unrelated Machines

Non-Clairvoyant Online Scheduling with Precedence Delays

Thank you!

II. Stochastic Scheduling

WSEPT Rule

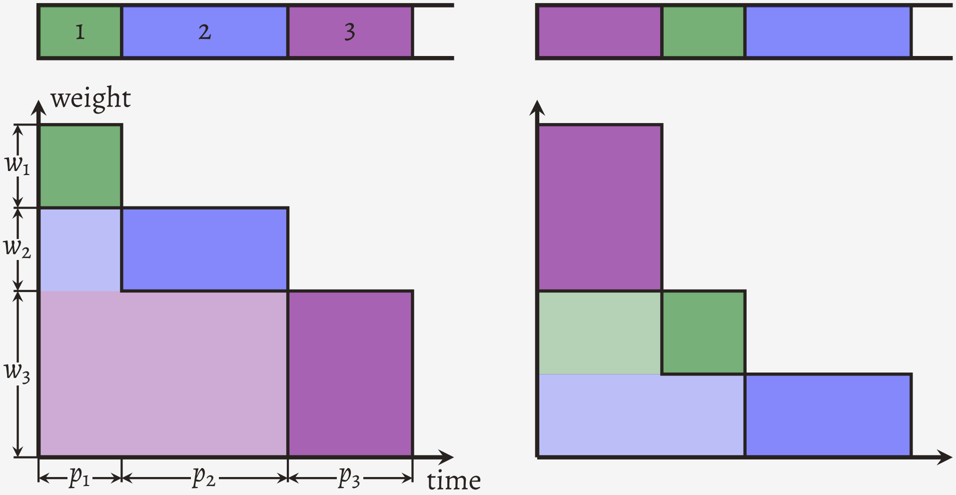
Stochastic Scheduling on Single Machine with Release Dates

Stochastic Online Scheduling on Unrelated Machines

Additional Slides

Two-Dimensional Gantt Charts

Eastman et al. 1964



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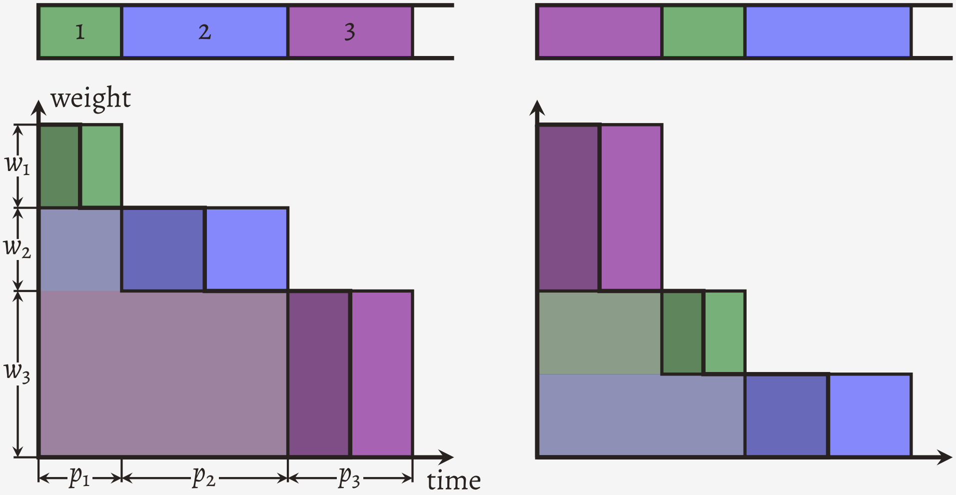

$$\sum_{j=1}^{n} w_j \cdot C_j = \sum_{j=1}^{n} w_j \cdot M_j + \frac{1}{2} \cdot \sum_{j=1}^{n} w_j \cdot p_j$$

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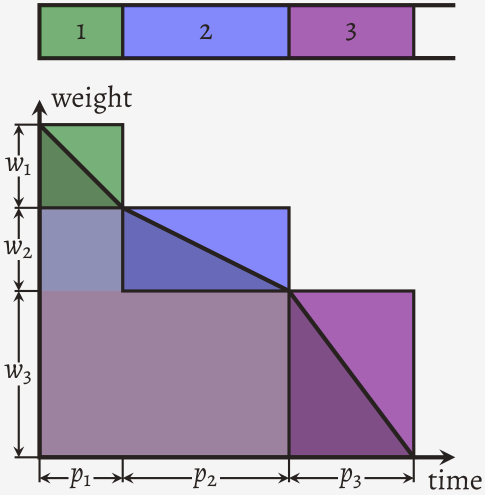

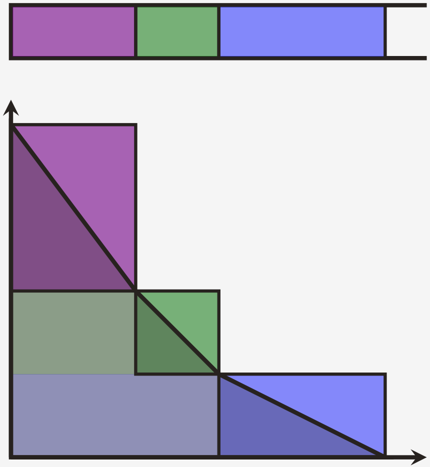

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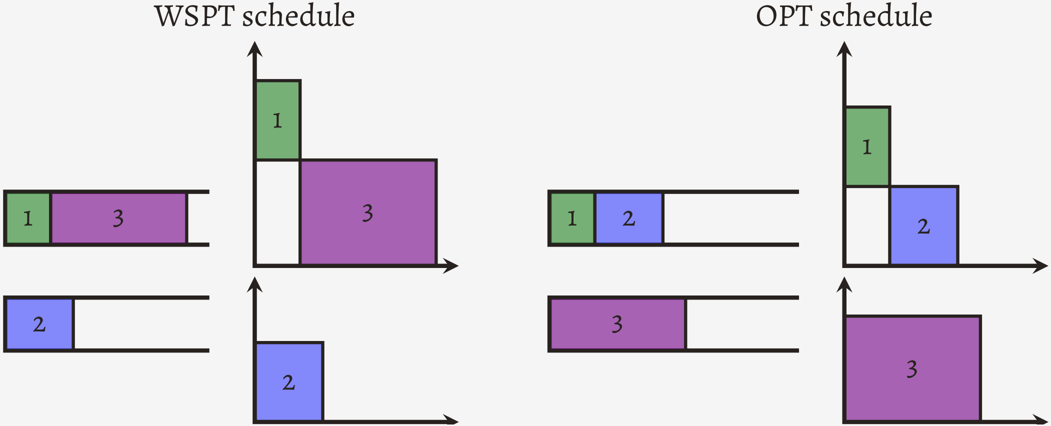
Reduction to Unit Smith Ratio

Lemma (Kawaguchi, Kyan 1986; Schwiegelshohn 2011)

On any number of machines the worst-case approximation ratio is attained when $w_j = p_j$ for all j.



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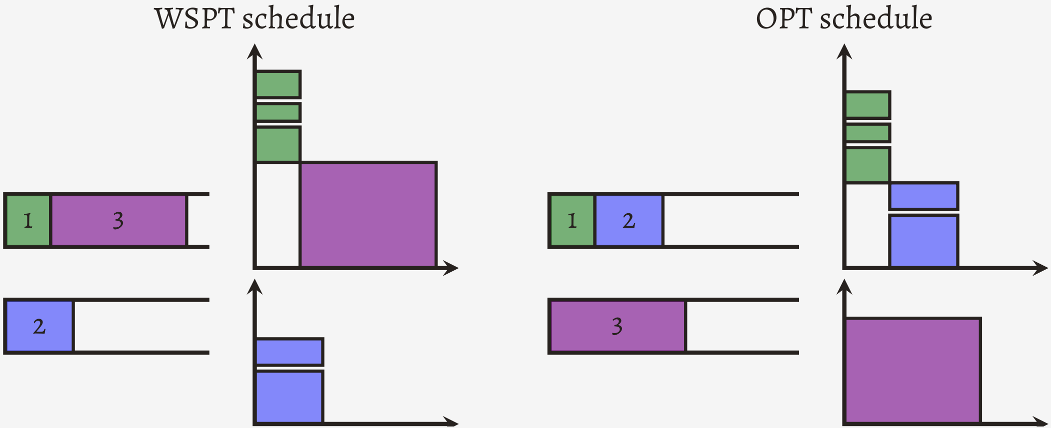

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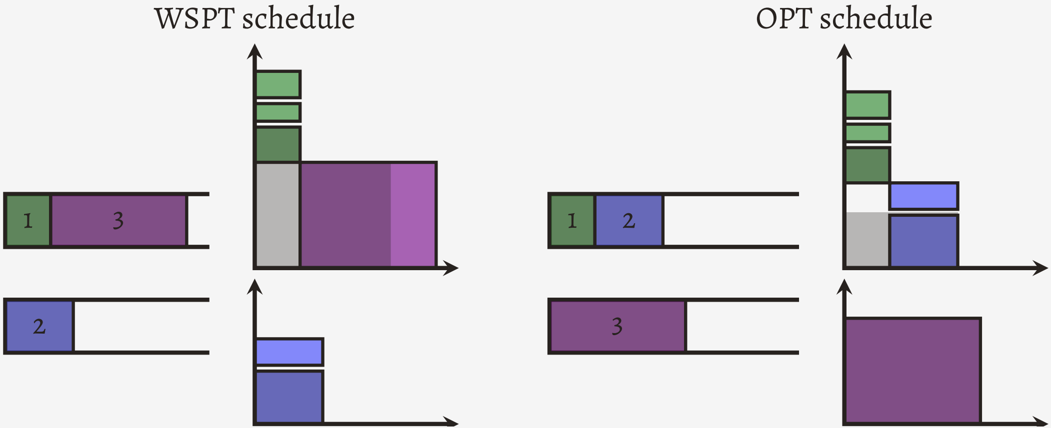

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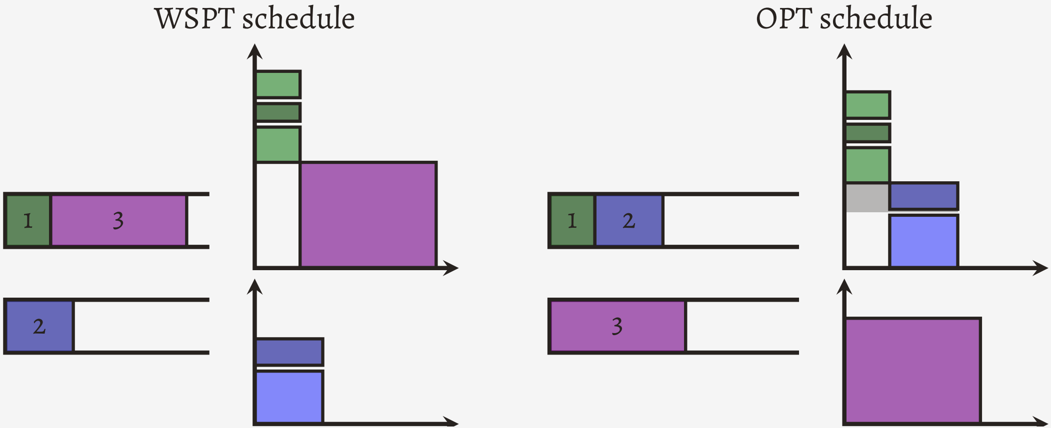

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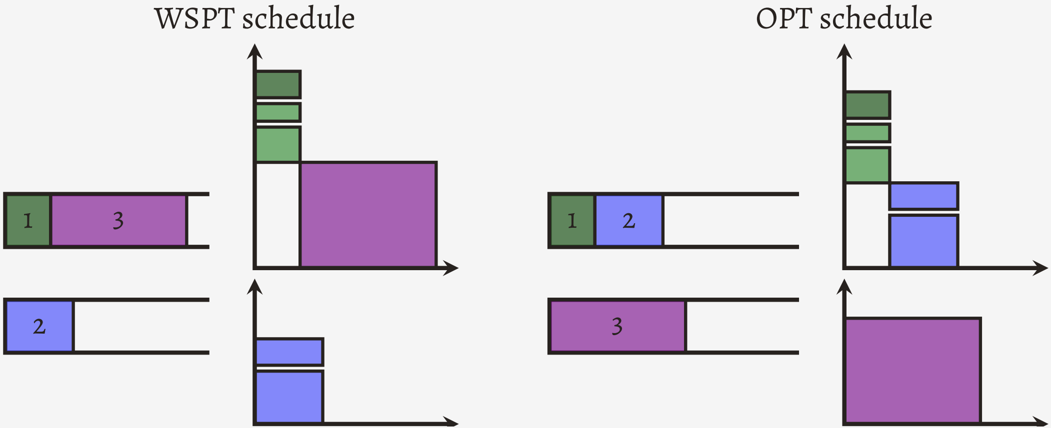

Approximation of the WSPT Rule for Sum of Weighted α-Points

For $\alpha \in (0, 1]$ the α-point $C_j(\alpha)$ of job j is the first moment when an α-fraction of j has been finished.

- $C_j(1) = C_j$
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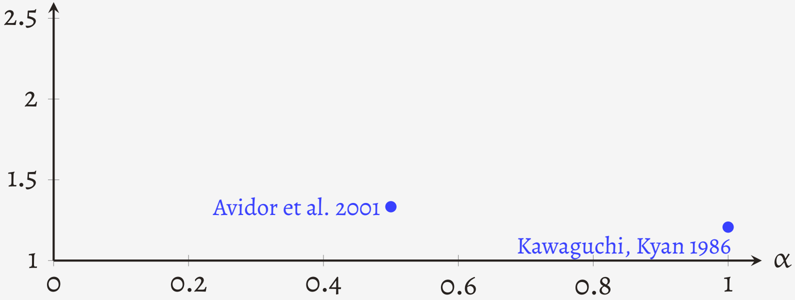

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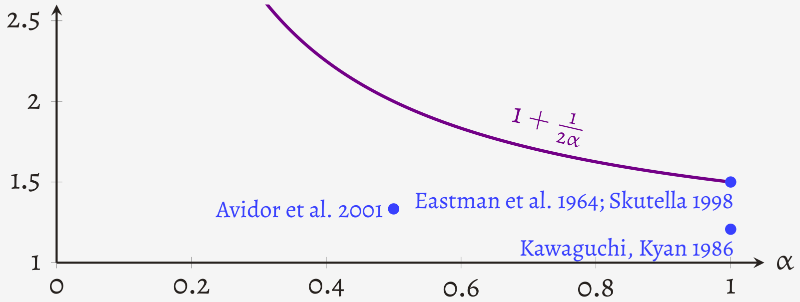

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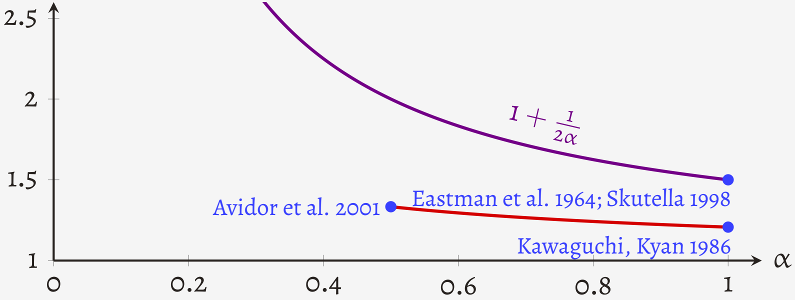

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For $\alpha \in \left[\frac{1}{2}, 1\right]$ the WSPT rule has approximation ratio

$$1 + \frac{1}{2\alpha + \sqrt{8\alpha}}.$$

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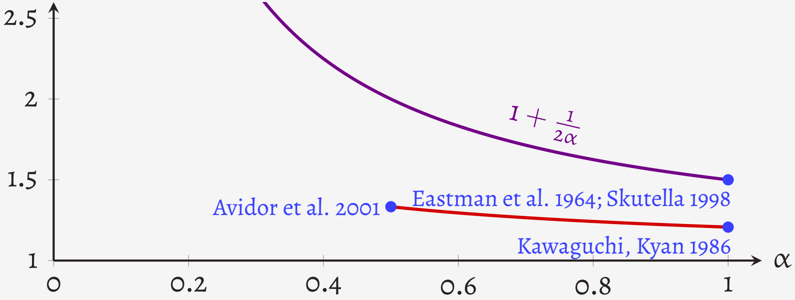

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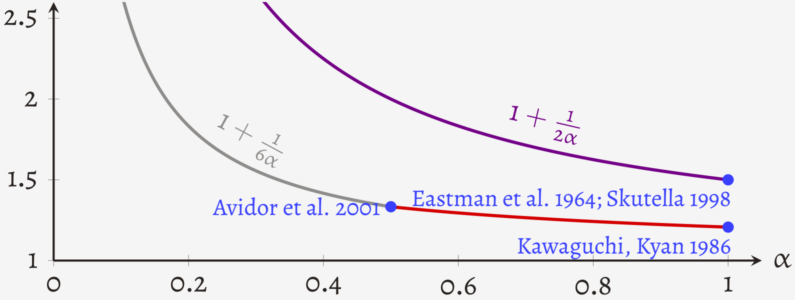

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Open Problem: Online List Assignment Model

- jobs arrive one by one; must be assigned to machines immediately
- on each machine, assigned jobs are optimally sequenced (WSPT)

Min-increase algorithm

Assign each job to a machine minimizing the increase of the current objective value.

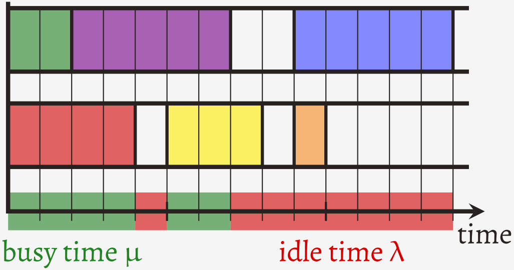
Known results:

- $\left(\frac{3}{2} - \frac{1}{2m}\right)$-competitive (cp. e.g. Megow et al. 2006)
- If jobs arrive in order of non-increasing or non-decreasing $\frac{w_j}{p_j}$, then Min-increase achieves competitive ratio $\frac{1+\sqrt{2}}{2}$.

Conjecture (Stougie 2017)

Min-increase has competitive ratio $\frac{1+\sqrt{2}}{2}$.

Scheduling with Precedence Delays on Identical Machines



Consider partial schedule when the blue job has been assigned.

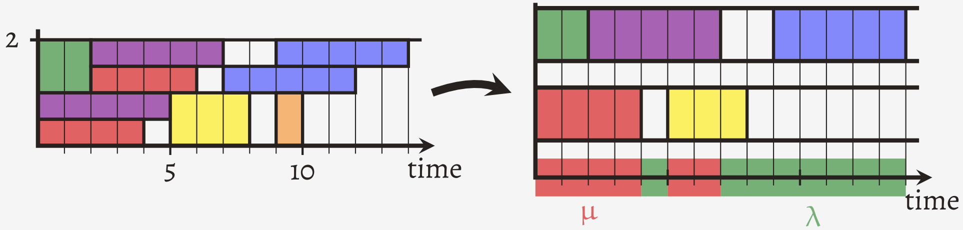
$$C_{\text{blue}} = \mu + \lambda.$$

For $\alpha \in (0, \frac{1}{2}]$ it holds that

$$\mu \leq \frac{1}{\alpha} \cdot C_{\text{blue}}^{\text{LP}}, \qquad \lambda \leq \frac{1}{1-\alpha} \cdot C_{\text{blue}}^{\text{LP}}.$$

For $\alpha = \frac{1}{2}$ this gives a 4-approximation algorithm.

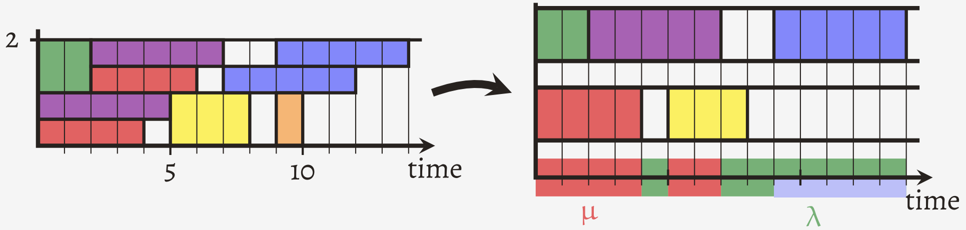
Bounding the Idle Time



Trace back why the blue job is not completed earlier.

$\lambda \quad \leq \dots$

Bounding the Idle Time

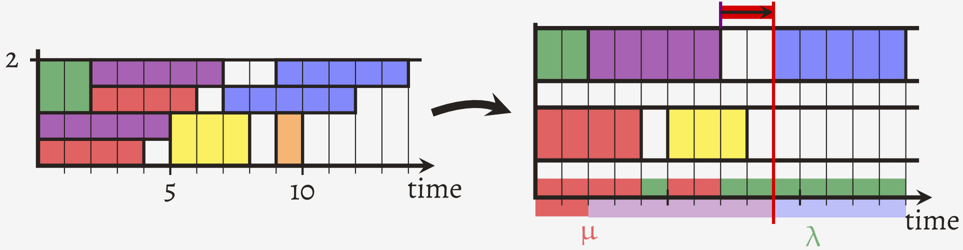


Trace back why the blue job is not completed earlier.

- The blue job has to be processed (can be idle time).

$$\lambda \leq p_{\text{blue}} + \dots$$

Bounding the Idle Time

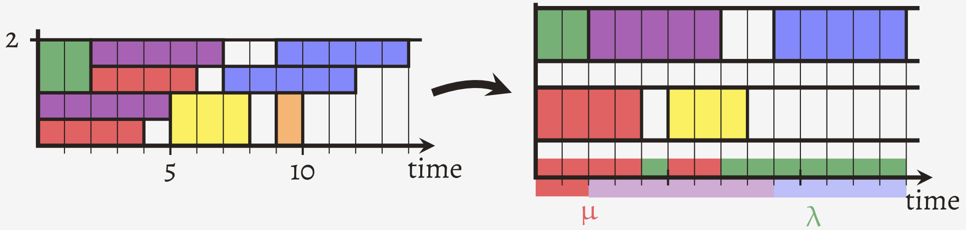


Trace back why the blue job is not completed earlier.

- The blue job has to be processed (can be idle time).
- The blue job has to wait for the purple job (can be idle time).

$$\lambda \leq p_{\text{blue}} + d_{\text{purple,blue}} + p_{\text{purple}}$$

Bounding the Idle Time



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- The blue job has to wait for the purple job (can be idle time).
- The purple job has to wait because before all machines are busy (cannot be idle time).

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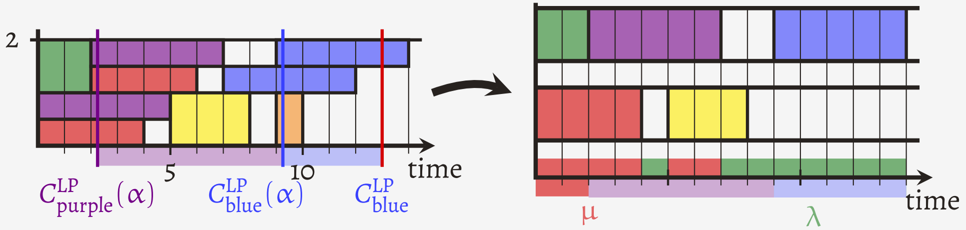

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$$\lambda \leq \underbrace{p_{blue}}_{=\frac{1}{1-\alpha}(C^{LP}_{blue} - C^{LP}_{blue}(\alpha))} + \underbrace{d_{purple,blue} + p_{purple}}_{\leq \frac{1}{1-\alpha}(C^{LP}_{blue}(\alpha) - C^{LP}_{purple}(\alpha))}$$

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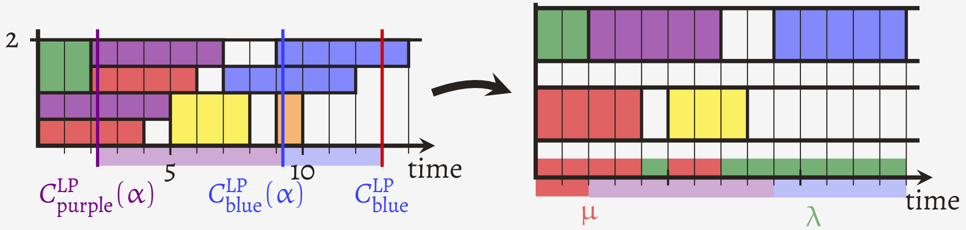

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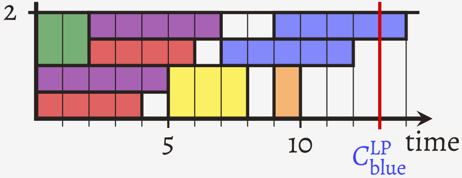
- The blue job has to wait for the purple job (can be idle time).
- The purple job has to wait because before all machines are busy (cannot be idle time).

$$\lambda \leq \underbrace{p_{blue}}_{=\frac{1}{1-\alpha}(C^{LP}_{blue}-C^{LP}_{blue}(\alpha))} + \underbrace{d_{purple,blue} + p_{purple}}_{\leq \frac{1}{1-\alpha}(C^{LP}_{blue}(\alpha)-C^{LP}_{purple}(\alpha))} \leq \frac{1}{1-\alpha} \cdot C^{LP}_{blue}.$$

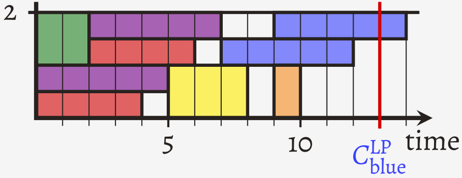
Bounding the Expected Busy Time



For $\alpha \in \left(0, \frac{1}{2}\right]$ let $J_\alpha := \left\{ j \in N \mid C_j^{LP}(\alpha) \le C^{LP}_{blue} \right\}$.

Upper bound on busy time: $\mu \le \dfrac{p(J_\alpha)}{m}$.

Bounding the Expected Busy Time

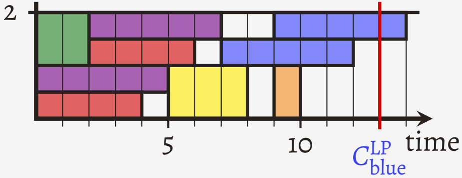


For $\alpha \in \left(0, \frac{1}{2}\right]$ let $J_\alpha := \left\{ j \in N \mid C^{LP}_j(\alpha) \le C^{LP}_{blue} \right\}$.

$$\text{Upper bound on busy time: } \mu \le \frac{p(J_\alpha)}{m}.$$

$$\text{Lower bound on LP-completion time: } C^{LP}_{blue} \ge \frac{E\left[p(J_\alpha)\right]}{2m}.$$

Bounding the Expected Busy Time



For $\alpha \in \left(0, \frac{1}{2}\right]$ let $J_\alpha := \left\{j \in N \mid C_j^{\text{LP}}(\alpha) \leq C^{\text{LP}}_{\text{blue}}\right\}$.

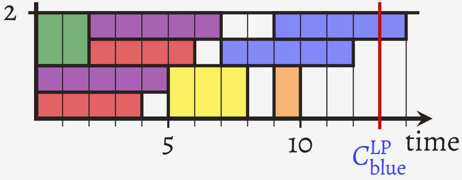
$$\text{Upper bound on busy time: } \mu \leq \frac{p(J_\alpha)}{m}.$$

$$\text{Lower bound on LP-completion time: } C^{\text{LP}}_{\text{blue}} \geq \frac{E\left[p(J_\alpha)\right]}{2m}.$$

$$E[\mu] \leq 2 \cdot C^{\text{LP}}_{\text{blue}}.$$

Bounding the Expected Busy Time

Lower bound on LP-completion time

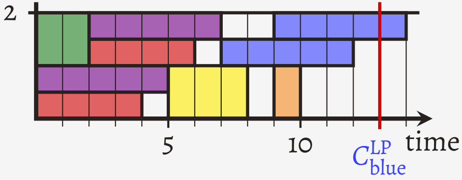


For $\alpha \in \left(0, \frac{1}{2}\right]$ let $J_\alpha := \left\{ j \in N \mid C_j^{LP}(\alpha) \leq C^{LP}_{blue} \right\}$.

Claim: $C^{LP}_{blue} \geq \frac{1}{2} \cdot \frac{E\left[p(J_\alpha)\right]}{m}$.

Bounding the Expected Busy Time

Lower bound on LP-completion time



For $\alpha \in (0, \frac{1}{2}]$ let $J_\alpha := \{j \in N \mid C_j^{LP}(\alpha) \le C^{LP}_{blue}\}$.

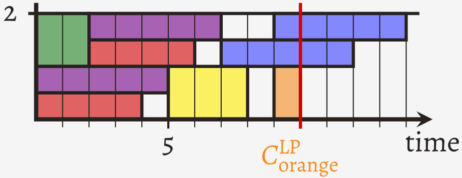
Claim: $C^{LP}_{blue} \ge \frac{1}{2} \cdot \frac{E[p(J_\alpha)]}{m}$.

In the example $C_j^{LP}(\frac{1}{2}) \le C^{LP}_{blue}$ for all $j \in N$, whence $J_\alpha = N$ for all α. Therefore,

$$C^{LP}_{blue} \ge \frac{1}{2} \cdot \frac{p(N)}{m} = \frac{1}{2} \cdot \frac{E[p(J_\alpha)]}{m}.$$

Bounding the Expected Busy Time

Lower bound on LP-completion time

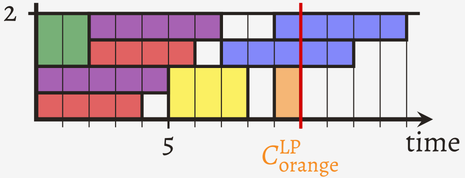


For $\alpha \in \left(0, \frac{1}{2}\right]$ let $J_\alpha := \left\{ j \in N \mid C_j^{\text{LP}}(\alpha) \le C^{\text{LP}}_{\text{orange}} \right\}$.

Claim: $C^{\text{LP}}_{\text{orange}} \ge \frac{1}{2} \cdot \frac{E\left[p(J_\alpha)\right]}{m}$.

Bounding the Expected Busy Time

Lower bound on LP-completion time



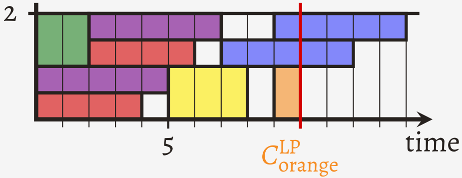
For $\alpha \in \left(0, \frac{1}{2}\right]$ let $J_\alpha \coloneqq \left\{ j \in N \mid C_j^{LP}(\alpha) \le C^{LP}_{\text{orange}} \right\}$.

Claim: $C^{LP}_{\text{orange}} \ge \frac{1}{2} \cdot \frac{E\left[p(J_\alpha)\right]}{m}$.

$$p(J_\alpha) = \begin{cases} p_{\text{green}} + p_{\text{purple}} + p_{\text{red}} + p_{\text{yellow}} + p_{\text{orange}} + p_{\text{blue}} & \text{if } \alpha \le \frac{2}{5}; \\ p_{\text{green}} + p_{\text{purple}} + p_{\text{red}} + p_{\text{yellow}} + p_{\text{orange}} & \text{if } \alpha > \frac{2}{5}. \end{cases}$$

Bounding the Expected Busy Time

Lower bound on LP-completion time



For $\alpha \in (0, \frac{1}{2}]$ let $J_\alpha := \{j \in N \mid C_j^{LP}(\alpha) \leq C^{LP}_{orange}\}$.

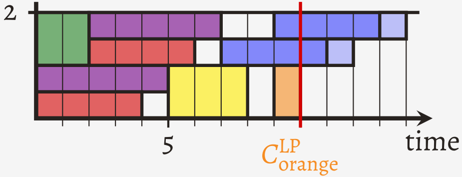
Claim: $C^{LP}_{orange} \geq \frac{1}{2} \cdot \frac{E[p(J_\alpha)]}{m}$.

$$p(J_\alpha) = \begin{cases} p_{green} + p_{purple} + p_{red} + p_{yellow} + p_{orange} + p_{blue} & \text{if } \alpha \leq \frac{2}{5}; \\ p_{green} + p_{purple} + p_{red} + p_{yellow} + p_{orange} & \text{if } \alpha > \frac{2}{5}. \end{cases}$$

$$E[p(J_\alpha)] = p_{green} + p_{purple} + p_{red} + p_{yellow} + p_{orange} + \frac{4}{5} \cdot p_{blue}$$

Bounding the Expected Busy Time

Lower bound on LP-completion time



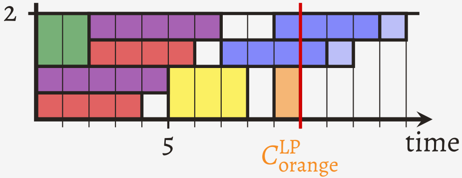
For $\alpha \in (0, \frac{1}{2}]$ let $J_\alpha := \{j \in N \mid C_j^{LP}(\alpha) \leq C^{LP}_{orange}\}$.

Claim: $C^{LP}_{orange} \geq \frac{1}{2} \cdot \frac{E[p(J_\alpha)]}{m}$.

$$E[p(J_\alpha)] = p_{green} + p_{purple} + p_{red} + p_{yellow} + p_{orange} + \frac{4}{5} \cdot p_{blue}$$

Bounding the Expected Busy Time

Lower bound on LP-completion time



For $\alpha \in \left(0, \frac{1}{2}\right]$ let $J_\alpha := \left\{ j \in N \mid C_j^{\text{LP}}(\alpha) \le C^{\text{LP}}_{\text{orange}} \right\}$.

Claim: $C^{\text{LP}}_{\text{orange}} \ge \frac{1}{2} \cdot \frac{E\left[p(J_\alpha)\right]}{m}$.

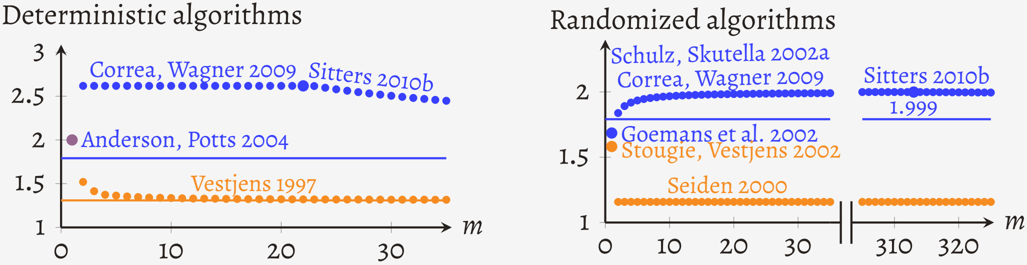
$$E[p(J_\alpha)] = p_{\text{green}} + p_{\text{purple}} + p_{\text{red}} + p_{\text{yellow}} + p_{\text{orange}} + \frac{4}{5} \cdot p_{\text{blue}}$$

The modified jobs J' have midpoints bounded by $C^{\text{LP}}_{\text{orange}}$. Therefore,

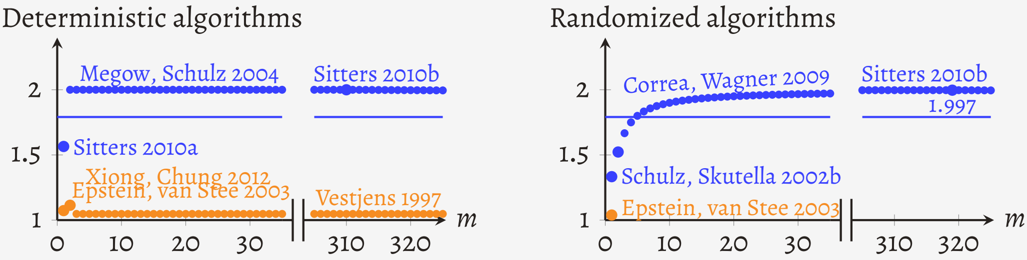
$$C^{\text{LP}}_{\text{orange}} \ge \frac{1}{2} \cdot \frac{p(J')}{m} = \frac{1}{2} \cdot \frac{E\left[p(J_\alpha)\right]}{m}.$$

Clairvoyant Online Time Model on Identical Machines

Non-preemptive scheduling



Preemptive scheduling



Non-Migratory Preemptive Scheduling on Unrelated Machines

Assume that all $p_{ij}, r_j \in 2\mathbb{Z}$, and let $T \in \mathbb{Z}$ be an upper bound on the makespan.

Variables: y_{ijt}, $i \in [m]$, $j \in [n]$, $t \in \{r_j, \dots, T-1\}$, indicating how long job j is processed on machine i during time slot $(t, t+1]$.

$$
\begin{aligned}
\text{(LP)}\ \min \quad & \sum_{j=1}^{n} w_j \cdot \sum_{i=1}^{m} \sum_{t=r_j}^{T-1} \left(\frac{y_{ijt}}{2} + \frac{y_{ijt}}{p_{ij}} \cdot \left(t + \frac{1}{2}\right) \right) \\
\text{s. t.} \quad & \sum_{i=1}^{m} \sum_{t=r_j}^{T-1} \frac{y_{ijt}}{p_{ij}} = 1 && \forall j \in [n] \\
& \sum_{j: r_j \geq t} y_{ijt} \leq 1 && \forall i \in [m],\ t \in \{0, \dots, T-1\} \\
& y_{ijt} \geq 0 && \forall i \in [m],\ j \in [n],\ t \in \{r_j, \dots, T-1\}
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\text{(LP)}\ \min\ & \sum_{j=1}^{n} w_j \cdot \sum_{i=1}^{m} \sum_{t=r_j}^{T-1} \left(\frac{y_{ijt}}{2} + \frac{y_{ijt}}{p_{ij}} \cdot \left(t + \frac{1}{2}\right) \right) \\
\text{s.t.}\ & \sum_{i=1}^{m} \sum_{t=r_j}^{T-1} \frac{y_{ijt}}{p_{ij}} = 1 && \forall j \in [n] \\
& \sum_{j: r_j \geq t} y_{ijt} \leq 1 && \forall i \in [m],\ t \in \{0, \ldots, T-1\} \\
& y_{ijt} \geq 0 && \forall i \in [m],\ j \in [n],\ t \in \{r_j, \ldots, T-1\}
\end{aligned}
$$

$$
\begin{aligned}
\text{(D)}\ \max\ & \sum_{j=1}^{n} \chi_j - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \psi_{it} \\
\text{s.t.}\ & \frac{\chi_j}{p_{ij}} \leq \psi_{it} + w_j \left(\frac{t + 1/2}{p_{ij}} + \frac{1}{2} \right) && \forall i \in [m],\ j \in [n],\ t \in \{r_j, \ldots, T-1\} \\
& \psi_{it} \geq 0 && \forall i \in [m],\ t \in \{0, \ldots, T-1\}
\end{aligned}
$$

Dual Solution

- For $j \in [n]$ let $i(j)$ be the machine to which j is assigned by the Greedy-Assignment WSRPT algorithm.
- For $i \in [m]$ and $t \in \mathbb{Z}_{\geq 0}$ let $U_{i,t}$ be the set of jobs completed after time t on machine i in the Greedy-Assignment WSRPT schedule.

$$\chi_j := \frac{1}{2} \cdot \text{cost}\big(j \to i(j)\big) \qquad \text{for all } j \in [n],$$

$$\psi_{it} := \frac{w(U_{i,2t})}{2} \qquad \text{for all } i \in [m],\ t \in \{0, \ldots, T-1\},$$

Cost of Dual Solution

$$
\begin{aligned}
\text{(D)}\ \max \quad & \sum_{j=1}^{n} \chi_j - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \psi_{it} \\
\text{s. t.} \quad & \frac{\chi_j}{p_{ij}} \leq \psi_{it} + w_j \cdot \left(\frac{t + 1/2}{p_{ij}} + \frac{1}{2} \right) && \forall i, j, t \\
& \psi_{it} \geq 0 && \forall i, t
\end{aligned}
$$

Cost of Dual Solution

$$(D)\ \max \quad \sum_{j=1}^{n} \frac{\text{cost}(j \to i(j))}{2} - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \frac{w(U_{i,2t})}{2}$$

$$\text{s. t.} \quad \frac{\chi_j}{p_{ij}} \leq \psi_{it} + w_j \cdot \left(\frac{t + 1/2}{p_{ij}} + \frac{1}{2} \right) \qquad \forall i, j, t$$

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Cost of Dual Solution

$$\sum_{j=1}^{n} \frac{\text{cost}\big(j \to i(j)\big)}{2} - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \frac{w(U_{i,2t})}{2}$$

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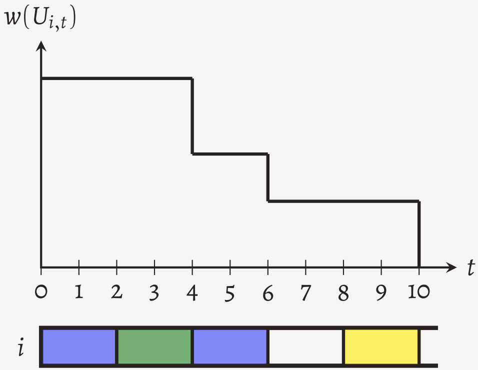
$$\sum_{j=1}^{n} \frac{\text{cost}(j \to i(j))}{2} - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \frac{w(U_{i,2t})}{2}$$

$$= \frac{1}{2} \cdot z^{\text{GA-WSRPT}} - \frac{1}{4} \cdot z^{\text{GA-WSRPT}} = \frac{1}{4} \cdot z^{\text{GA-WSRPT}}$$



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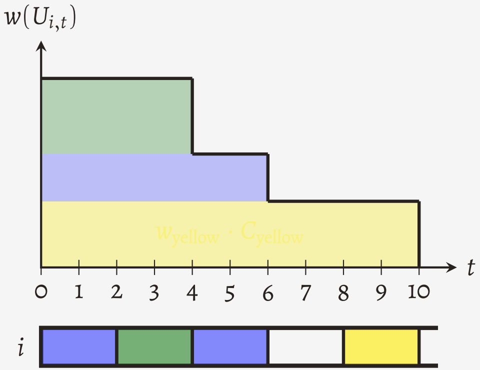

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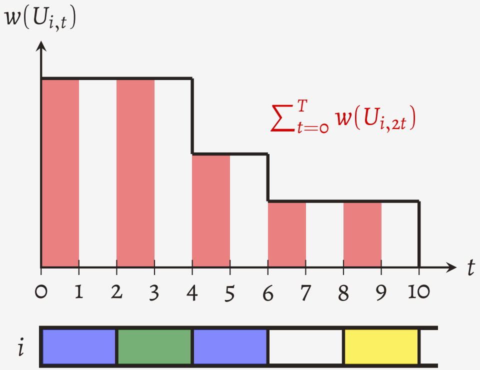


Feasibility of Dual Solution

$$\text{(D)} \quad \max \quad \sum_{j=1}^{n} \chi_j - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \psi_{it}$$

$$\text{s.t.} \quad \frac{\chi_j}{p_{ij}} \leq \psi_{it} + w_j \cdot \left(\frac{t + 1/2}{p_{ij}} + \frac{1}{2} \right) \quad \forall i, j, t$$

$$\psi_{it} \geq 0 \quad \forall i, t$$

Feasibility of Dual Solution

$$
\begin{aligned}
\text{(D)}\ \max\quad & \sum_{j=1}^{n} \chi_j - \sum_{i=1}^{m} \sum_{t=0}^{T-1} \psi_{it} \\
\text{s. t.}\quad & \frac{\mathrm{cost}(j \to i(j))}{2 \cdot p_{ij}} \le \frac{w(U_{i,2t})}{2} + w_j \cdot \left(\frac{t + 1/2}{p_{ij}} + \frac{1}{2}\right) && \forall i, j, t \\
& \frac{w(U_{i,2t})}{2} \ge 0 && \forall i, t
\end{aligned}
$$

Feasibility of Dual Solution

$$\frac{\text{cost}(j \rightarrow i(j))}{2 \cdot p_{ij}} \leq \frac{w(U_{i,2t})}{2} + w_j \cdot \left(\frac{t + 1/2}{p_{ij}} + \frac{1}{2} \right) \quad \forall i, j, t$$

Weighted Shortest Expected Processing Time First Rule

WSEPT rule

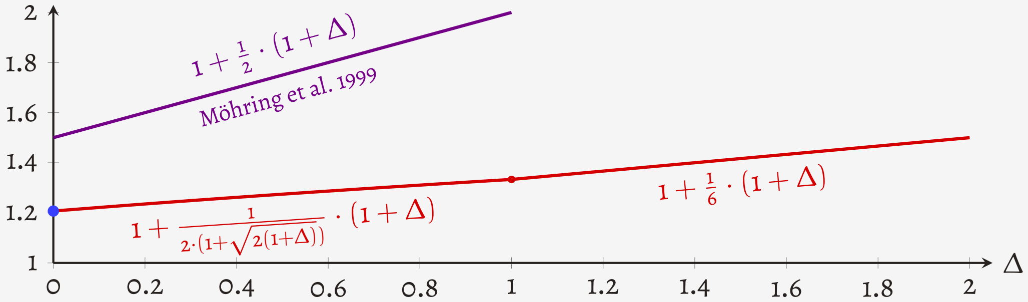
Whenever a machine is free, start available job with maximum ratio $w_j / E[\boldsymbol{p}_j]$ on it.

- WSEPT is optimal if
  - there is only one machine (Rothkopf 1966)
  - all jobs have unit weight, and the processing times are pairwise stochastically comparable (Weber et al. 1986)
- Even for unit-weight jobs WSEPT has no constant performance guarantee. (Cheung et al. 2014; Im et al. 2015)
- The approximation ratio can be bounded in terms of

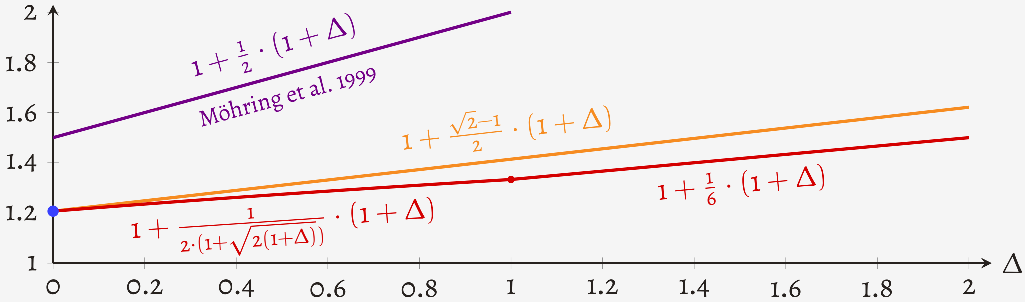
$$\Delta := \max_{j \in \{1,\dots,n\}} \frac{\mathrm{Var}[\boldsymbol{p}_j]}{E[\boldsymbol{p}_j]^2}.$$

(Möhring et al. 1999)

Performance Guarantees for the WSEPT Rule



Performance Guarantees for the WSEPT Rule



Proof of the Performance Guarantee

Idea: Consider random weights $\boldsymbol{w}_j' := \frac{\boldsymbol{p}_j}{E[\boldsymbol{p}_j]} \cdot w_j$.

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$\implies$ in every realization the WSEPT rule schedules the jobs in non-increasing order of $\frac{\mathbf{w}_j'}{\mathbf{p}_j}$.

$$\sum_{j=1}^{n} \mathbf{w}_j' \cdot \mathbf{C}_j^{\mathrm{WSEPT}} \leq \frac{1+\sqrt{2}}{2} \cdot \sum_{j=1}^{n} \mathbf{w}_j' \cdot \mathbf{C}_j^{\mathrm{OPT}(\mathbf{p},\mathbf{w}')} \leq \frac{1+\sqrt{2}}{2} \cdot \sum_{j=1}^{n} \mathbf{w}_j' \cdot \mathbf{C}_j^{\mathrm{OPT}}.$$

Proof of the Performance Guarantee

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Lemma

For any stochastic scheduling policy Π it holds that

$$E\left[\sum_{j=1}^{n} \boldsymbol{w}_j' \cdot \boldsymbol{C}_j^{\Pi}\right] = E\left[\sum_{j=1}^{n} w_j \cdot \boldsymbol{C}_j^{\Pi}\right] + \underbrace{\sum_{j=1}^{n} w_j \cdot \frac{\mathrm{Var}[\boldsymbol{p}_j]}{E[\boldsymbol{p}_j]}}_{=:c}.$$

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Proof

$$E[\boldsymbol{p}_j \boldsymbol{C}_j^{\Pi}] = E[\boldsymbol{p}_j \boldsymbol{S}_j] + E[\boldsymbol{p}_j^2] = E[\boldsymbol{p}_j] \cdot \big(E[\boldsymbol{S}_j] + E[\boldsymbol{p}_j]\big) + \text{Var}[\boldsymbol{p}_j] = E[\boldsymbol{p}_j] \cdot E[\boldsymbol{C}_j] + \text{Var}[\boldsymbol{p}_j]. \quad \square$$

Proof of the Performance Guarantee

Idea: Consider random weights $\boldsymbol{w}_j' := \frac{\boldsymbol{p}_j}{\mathrm{E}[\boldsymbol{p}_j]} \cdot w_j$.

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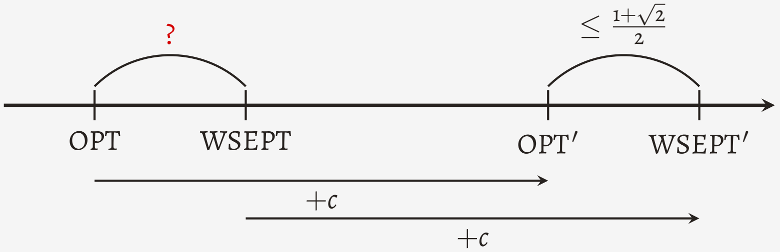
Proof

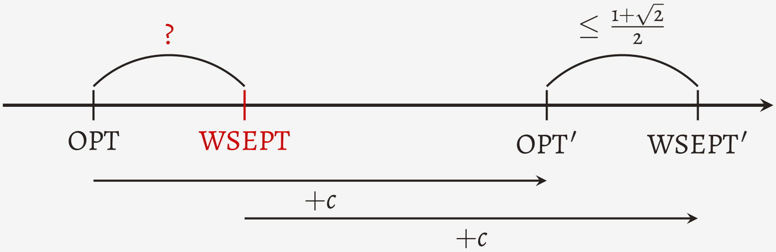
$$\mathrm{E}[\boldsymbol{p}_j \boldsymbol{C}_j^{\Pi}] = \mathrm{E}[\boldsymbol{p}_j \boldsymbol{S}_j] + \mathrm{E}[\boldsymbol{p}_j^2] = \mathrm{E}[\boldsymbol{p}_j] \cdot \big(\mathrm{E}[\boldsymbol{S}_j] + \mathrm{E}[\boldsymbol{p}_j]\big) + \mathrm{Var}[\boldsymbol{p}_j] = \mathrm{E}[\boldsymbol{p}_j] \cdot \mathrm{E}[\boldsymbol{C}_j] + \mathrm{Var}[\boldsymbol{p}_j]. \quad \square$$

Proof of the Performance Guarantee (Cont.)



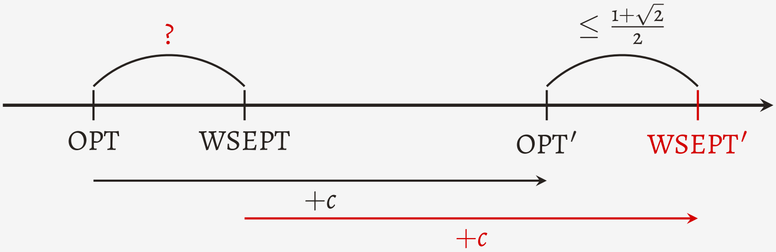
WSEPT

Proof of the Performance Guarantee (Cont.)



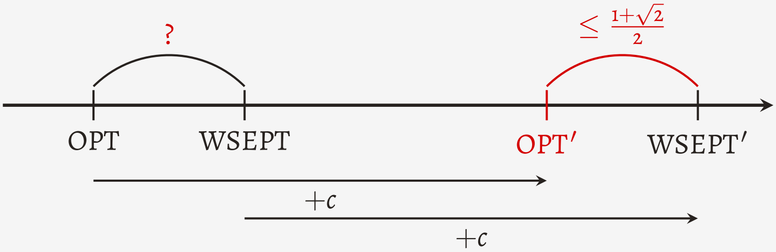
WSEPT

Proof of the Performance Guarantee (Cont.)



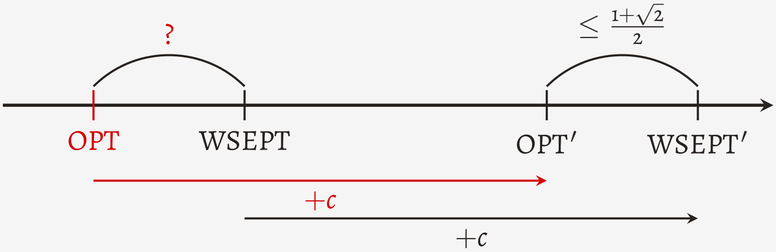
$$\text{WSEPT} = \text{WSEPT}' - c$$

Proof of the Performance Guarantee (Cont.)



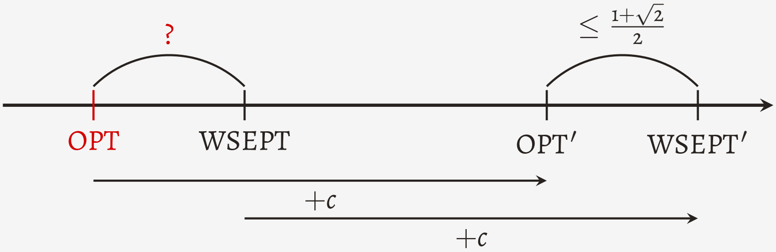
$$\mathrm{WSEPT} = \mathrm{WSEPT}' - c \leq \frac{1+\sqrt{2}}{2} \cdot \mathrm{OPT}' - c$$

Proof of the Performance Guarantee (Cont.)



$$\text{WSEPT} = \text{WSEPT}' - c \leq \frac{1+\sqrt{2}}{2} \cdot \text{OPT}' - c = \frac{1+\sqrt{2}}{2} \cdot (\text{OPT} + c) - c$$

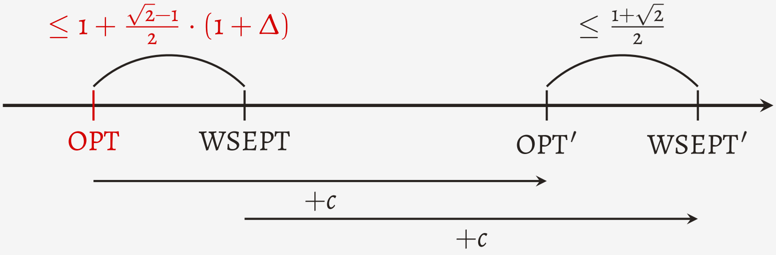
Proof of the Performance Guarantee (Cont.)



$$\text{WSEPT} = \text{WSEPT}' - c \leq \frac{1+\sqrt{2}}{2} \cdot \text{OPT}' - c = \frac{1+\sqrt{2}}{2} \cdot (\text{OPT} + c) - c$$

$$= \text{OPT} + \frac{\sqrt{2}-1}{2} \cdot (\text{OPT} + c)$$

Proof of the Performance Guarantee (Cont.)

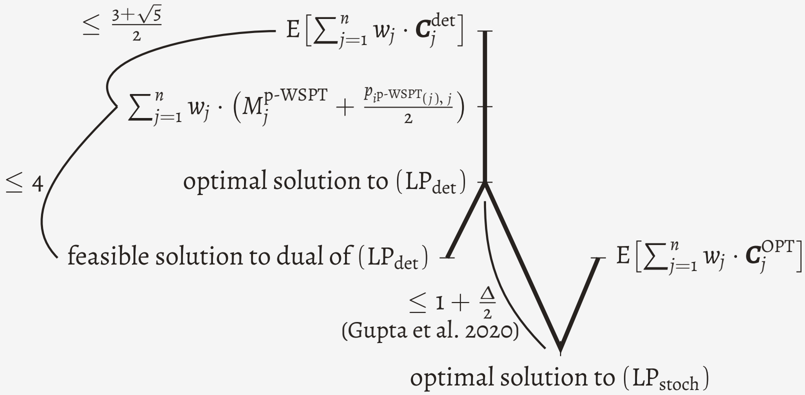


$$\text{WSEPT} = \text{WSEPT}' - c \leq \frac{1+\sqrt{2}}{2} \cdot \text{OPT}' - c = \frac{1+\sqrt{2}}{2} \cdot (\text{OPT} + c) - c$$

$$= \text{OPT} + \frac{\sqrt{2}-1}{2} \cdot (\text{OPT} + c) \leq \left(1 + \frac{\sqrt{2}-1}{2} \cdot (1+\Delta)\right) \cdot \text{OPT}$$

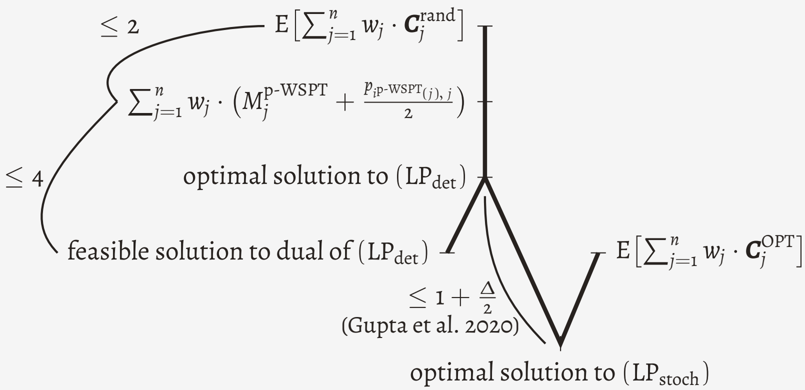
$c \leq \Delta\text{OPT}$

Analysis of Online Scheduling Policies for Unrelated Machines



deterministic online $\quad \frac{3+\sqrt{5}}{2} \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = (3+\sqrt{5}) \cdot (2+\Delta)$

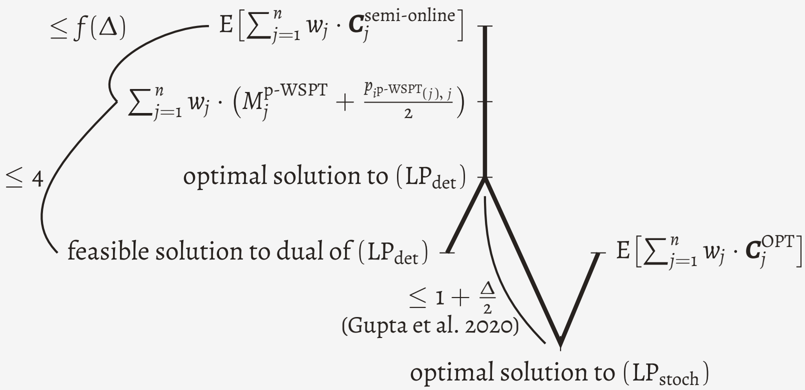
Analysis of Online Scheduling Policies for Unrelated Machines



deterministic online $\quad \frac{3+\sqrt{5}}{2} \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = (3 + \sqrt{5}) \cdot (2 + \Delta)$

randomized online $\quad 2 \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = 4 \cdot (2 + \Delta)$

Analysis of Online Scheduling Policies for Unrelated Machines

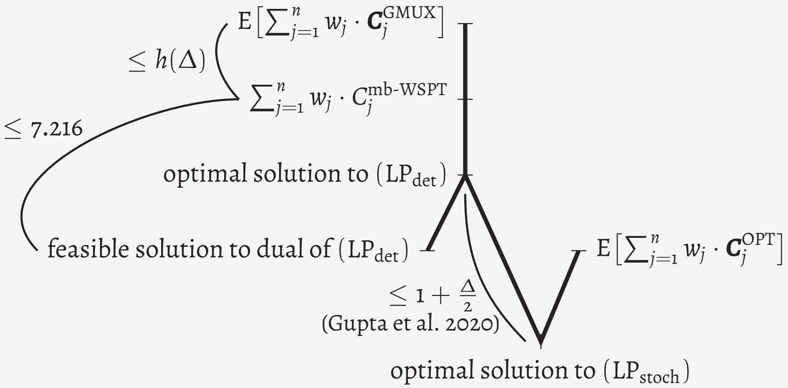


deterministic online $\quad \frac{3+\sqrt{5}}{2} \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = (3+\sqrt{5}) \cdot (2+\Delta)$

randomized online $\quad 2 \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = 4 \cdot (2+\Delta)$

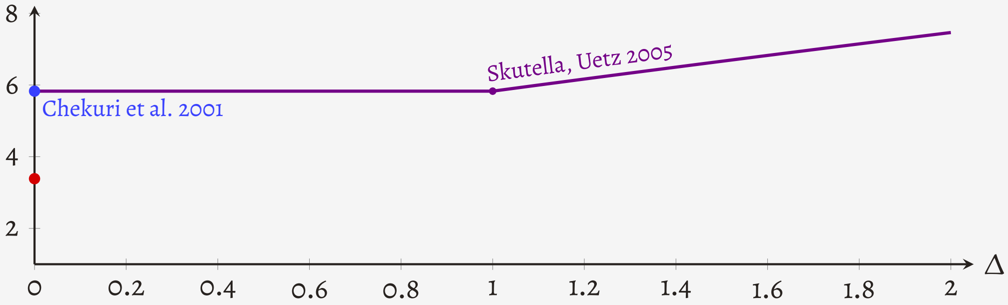
semi-online $\quad f(\Delta) \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = f(\Delta) \cdot 2 \cdot (2+\Delta)$

Analysis of Online Scheduling Policies for Unrelated Machines



deterministic online	$\frac{3+\sqrt{5}}{2} \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = (3+\sqrt{5}) \cdot (2+\Delta)$
randomized online	$2 \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = 4 \cdot (2+\Delta)$
semi-online	$f(\Delta) \cdot 4 \cdot \left(1 + \frac{\Delta}{2}\right) = f(\Delta) \cdot 2 \cdot (2+\Delta)$
Gupta et al. 2021+	$h(\Delta) \cdot 7.216 \cdot \left(1 + \frac{\Delta}{2}\right) = 3.608 \cdot h(\Delta) \cdot (2+\Delta)$

Stochastic Scheduling with Precedence Constraints and Release Dates



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