On orthogonal symmetric chain decompositions

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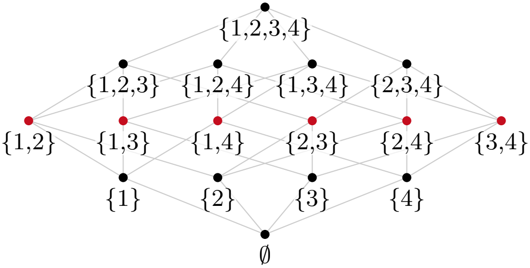
The n-cube

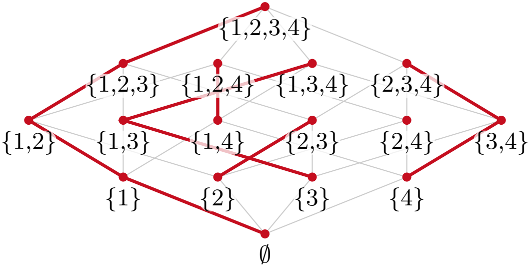
n-**cube** Q_n**:** all subsets of $[n] := \{1, \dots, n\}$ ordered by inclusion

k-**th level:** all subsets of cardinality k



Chain decompositions

- The n-cube can be decomposed into $\mathrm{width}(Q_n) = \binom{n}{\lfloor n/2 \rfloor}$ many chains. **[Sperner '28, Dilworth '50]**
- In this talk: **chain decompositon** = decompsition into $\mathrm{width}(Q_n)$ many chains.



Orthogonal chain decompositions

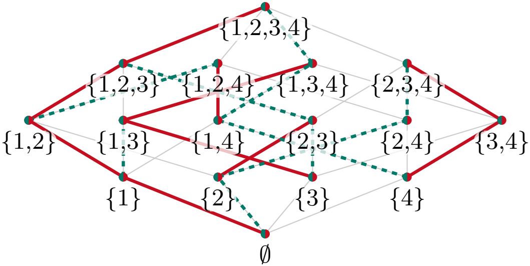
Two chain decompositions are **orthogonal** if every two chains have at most one element in common.

Theorem (Shearer, Kleitman '79)

The n-cube has two orthogonal chain decompositions for $n \geq 2$.

Conjecture (Shearer, Kleitman '79)

The n-cube has $\lfloor \frac{n}{2} \rfloor + 1$ pairwise orthogonal chain decompositions.



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Theorem (Spink '19)

The n-cube has three orthogonal chain decompositions for $n \geq 24$.

Theorem (our main result)

The n-cube has four orthogonal chain decompositions for $n \geq 60$.

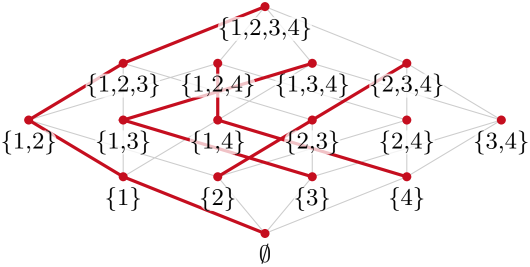
Symmetric chain decompositions

A chain is called

- **saturated** if it does not skip intermediate levels;
- **symmetric** if it starts at level k and ends at level $n-k$ for some $k \in \{0, \ldots, n\}$.

Theorem (de Bruijn, van Ebbenhorst-Tengbergen, Kruiswijk '51)

The n-cube can be decomposed into saturated, symmetric chains.



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Theorem (de Bruijn, van Ebbenhorst-Tengbergen, Kruiswijk '51)

The n-cube can be decomposed into saturated, symmetric chains.

- A decomposition into saturated, symmetric chains is called **symmetric chain decomposition** (**SCD**).
- The decomposition of de Bruijn et al. is called **standard SCD**.

Two orthogonal chain decompositions

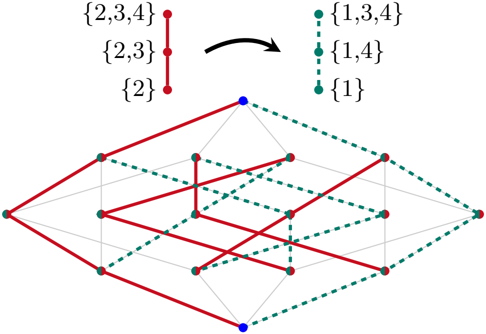
Theorem (Shearer, Kleitman '79)

The n-cube has two orthogonal chain decompositions for $n \geq 2$.

Proof idea

- Take standard SCD and its **complement SCD**.

$n = 4:$



- Only the two longest chains have two elements in common.

Two orthogonal chain decompositions

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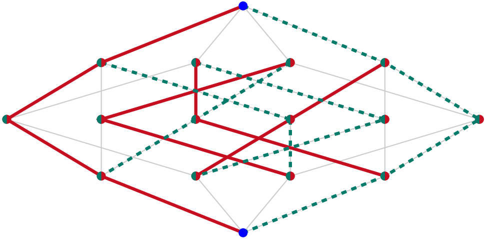
- Take standard SCD and its **complement SCD**.



- Only the two longest chains have two elements in common. Such SCDs are called **almost orthogonal**.
- Move $\emptyset$ in one decomposition to a shortest chain. □

Almost orthogonal SCDs

Generalization (Spink '19): If Q_n has k almost orthogonal SCDs, then Q_n has k orthogonal chain decompositions.

Proof: There are enough available shortest chains. □

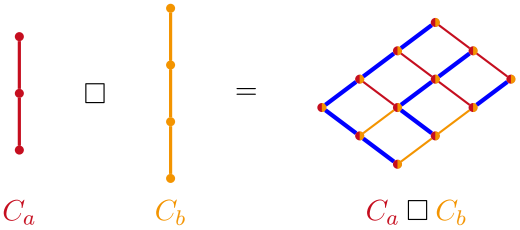
Proposition

The n-cube has four almost orthogonal SCDs for $n \geq 60$.

Why almost orthogonal SCDs?

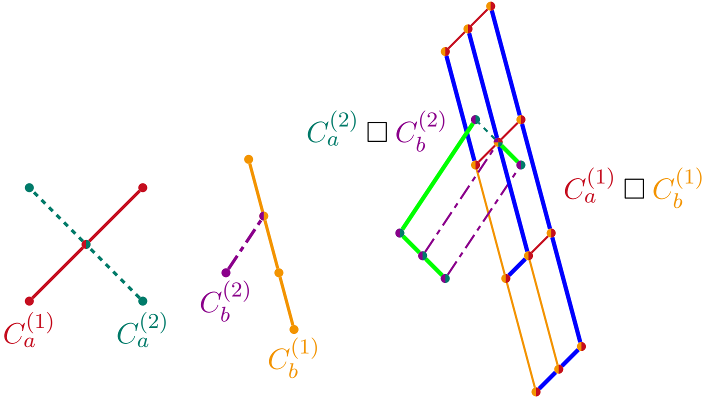
- much stronger requirements (symmetry, saturatedness)
- allow for product constructions

Product of chain decompositions



- If C_a and C_b are symmetric, saturated chains in Q_a and Q_b, then the chains in $C_a \square C_b$ are symmetric, saturated chains in Q_{a+b}.
- The **Cartesian product** $\mathcal{D}_a \square \mathcal{D}_b$ of chain decompositions $\mathcal{D}_a$ and $\mathcal{D}_b$ consists of the chains in $C_a \square C_b$ for all pairs $(C_a, C_b) \in \mathcal{D}_a \times \mathcal{D}_b$.

Products of orthogonal chains



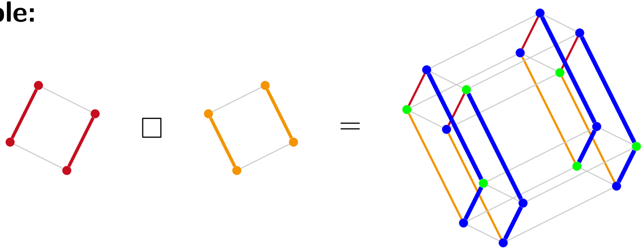
If $\left|C_a^{(1)} \cap C_a^{(2)}\right| \leq 1$ and $\left|C_b^{(1)} \cap C_b^{(2)}\right| \leq 1$, then

$$\left|(C_a^{(1)} \square C_b^{(1)}) \cap (C_a^{(2)} \square C_b^{(2)})\right| \leq 1.$$

Product of non-symmetric chain decompositions

Problem: The Cartesian product of non-symmetric chain decompositions can have more than $\mathrm{width}(Q_{a+b})$ chains!

Example:



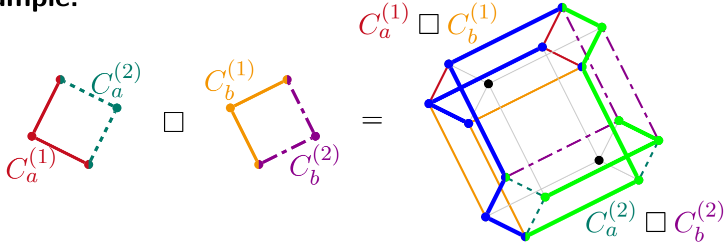
$\rightsquigarrow$ eight chains, but $\mathrm{width}(Q_4) = 6$.

Solution: Use symmetric chain decompositions. 😌

Products of two almost orthogonal SCDs

Problem: Chains in the chain product of longest chains may intersect too often.

Example:



Idea: Decompose one chain product differently.

- More complicated for more than two almost orthogonal SCDs.

Products of more than two almost orthogonal SCDs

Theorem (Spink '19)

Let $r \geq 6$, $n_1, \ldots, n_r \geq 5$ odd. If each Q_{n_i}, $i \in [r]$, has k almost orthogonal SCDs, then $Q_{n_1+\cdots+n_r}$ has k almost orthogonal SCDs.

$\Rightarrow$ Enough to find almost orthogonal SCDs in n-cubes for small n.

Lemma

Q_7 and Q_{11} have four almost orthogonal SCDs.

Proposition

The n-cube has four almost orthogonal SCDs for $n \geq 60$.

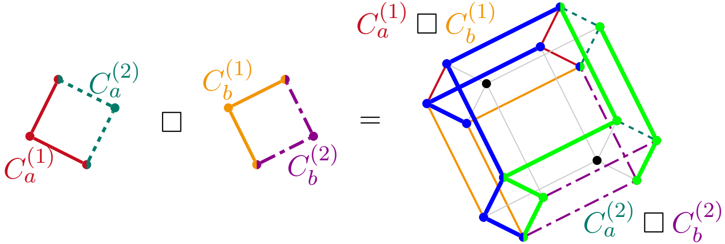
Proof: Every $n \geq 60$ can be written as $a \cdot 7 + b \cdot 11$ with $a + b \geq 6$, so that the result follows from Spink's theorem. □

Weaker notion: edge-disjoint SCDs

Two SCDs of the n-cube are **edge-disjoint** if no two chains have two **consecutive** elements in common.

Proposition (Gregor, J., Mütze, Sawada, Wille '18)

If $\mathcal{D}_a^{(1)}$ and $\mathcal{D}_a^{(2)}$ are edge-disjoint and $\mathcal{D}_b^{(2)}$ are edge-disjoint, then $\mathcal{D}_a^{(1)} \square \mathcal{D}_b^{(1)}$ and $\mathcal{D}_a^{(2)} \square \mathcal{D}_b^{(2)}$ are edge-disjoint.



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Lemma

Q_{10} and Q_{11} have five edge-disjoint SCDs.

Theorem

The n-cube has five edge-disjoint SCDs for $n \geq 90$.

Proof: Every $n \geq 90$ can be written as $a \cdot 10 + b \cdot 11$. $\square$

SCDs in small dimensions

Problem: Even for small n naive SAT formulation is too large. 😟

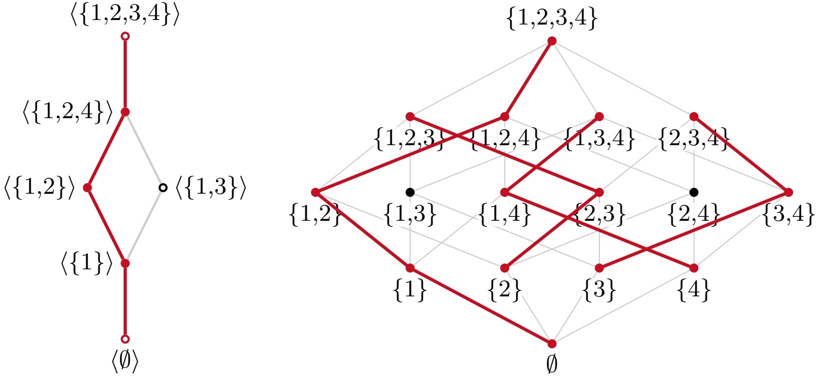
Goal: Reduce search space.

- $x, y \in Q_n$ are equivalent if they result from each other by a cyclic renaming $x \mapsto (x+1) \bmod n$.
- Equivalence classes are called **necklaces**, and quotient poset N_n is called **necklace poset**.
- A necklace $\langle x \rangle$ is **full** if $|\langle x \rangle| = n$, and **deficient** otherwise.

Example: $n = 4$.

- $\big\langle\{1,3,4\}\big\rangle = \big\{\{1,3,4\}, \{2,4,1\}, \{3,1,2\}, \{4,2,3\}\big\}$ full
- ◦ $\big\langle\{1,3\}\big\rangle = \big\{\{1,3\}, \{2,4\}\big\}$ deficient

Unrolling symmetric chains of necklaces



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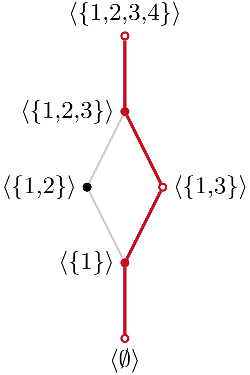
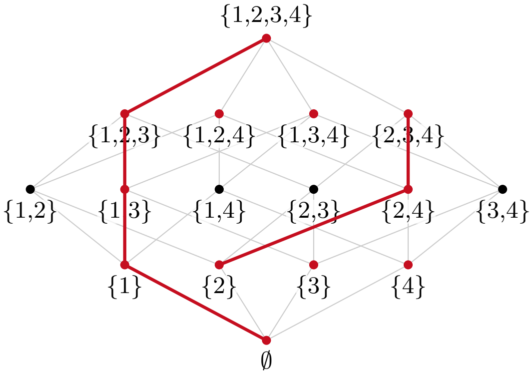
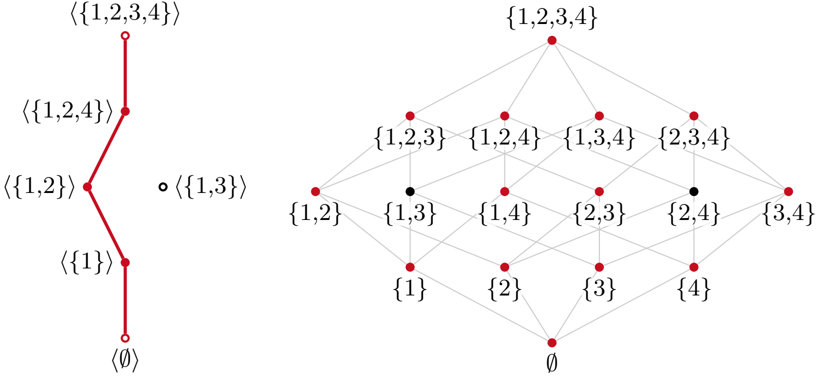




A chain in N_n is **unimodal** if its minimal and maximal necklace have the same size and all its other necklaces are full.

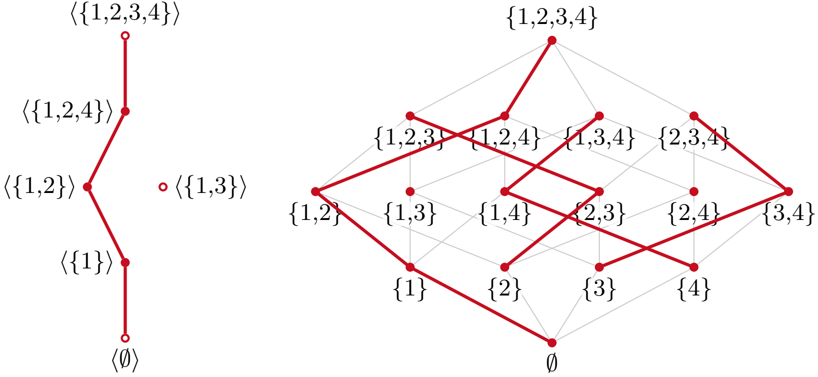
A unimodal chain in N_n can be unrolled to chains in Q_n.

Unrolling symmetric chains of necklaces



When looking for unimodal chains, we can remove some edges.

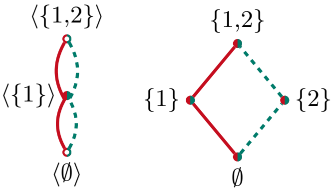
Unrolling SCDs of necklace poset



An SCD of N_n is **unimodal** if all its chains are unimodal.

Unimodal SCDs in N_n can be unrolled to SCDs in Q_n.

Unrolling multiple SCDs

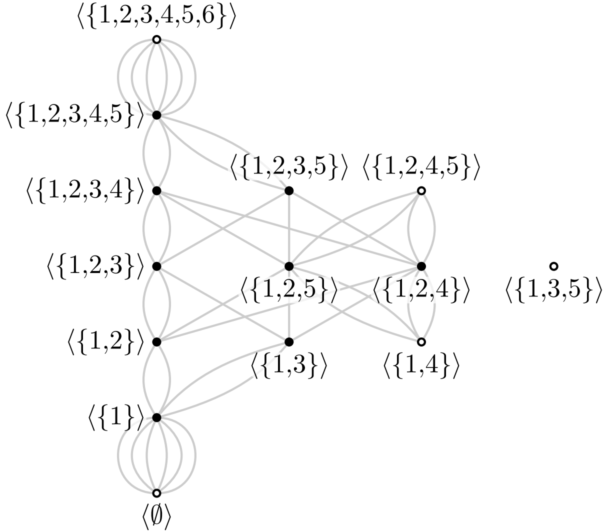


- Use multiple edges to represent multiple possibilities to go from one necklace to another.
- Find edge-disjoint unimodal SCDs in the multigraph and try to unroll them to almost orthogonal/edge-disjoint SCDs in Q_n.
- If all necklaces are full, edge-disjoint chains in the multigraph yield edge-disjoint chains in Q_n.

Multigraph

Example

$n = 6.$



Using SAT solvers

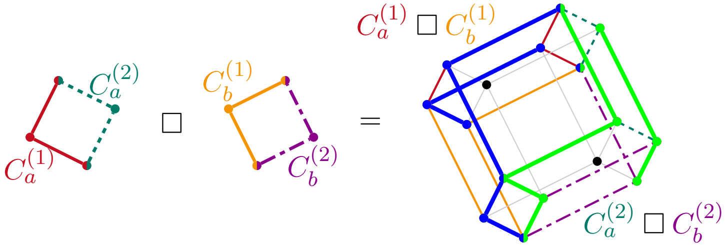
Formulate problem to find multiple edge-disjoint SCDs in multigraph as propositional formula in conjunctive normal form.

Result: Edge-disjoint SCDs in the multigraph.

- If not unrollable to edge-disjoint/almost orthogonal SCDs of Q_n, add clause forbidding particular configuration.
- Incremental SAT solver can reuse structural information about formula after adding new clauses.

Concluding remarks

- To help SAT solver prescribe some known unimodal SCDs of N_n (cf. **Griggs, Killian, Savage '04**; **Jordan '10**) that can be unrolled to almost-orthogonal/edge-disjoint SCDs.
- Obtained families of SCDs and independent verification program are available online.
- n-cube has four almost orthogonal SCDs for many $n < 60$.
- Q_n has three almost orthogonal SCDs $\iff n \geq 5$.
  Q_n has four edge-disjoint SCDs $\iff n \geq 6$.



Thank you!

Literature

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Additional Slides

Proof of upper bound on number of orthogonal chain decompositions

Edge-disjoint SCDs that are not unrollable to edge-disjoint SCDs

Numbers of SCDs in small dimensions

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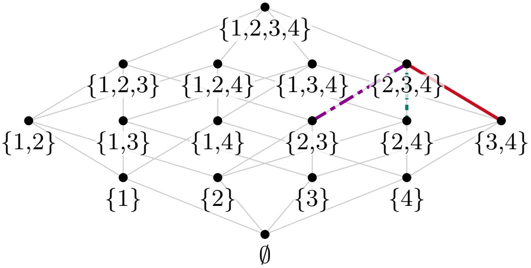
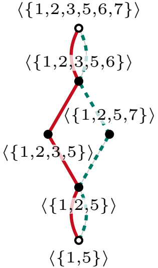
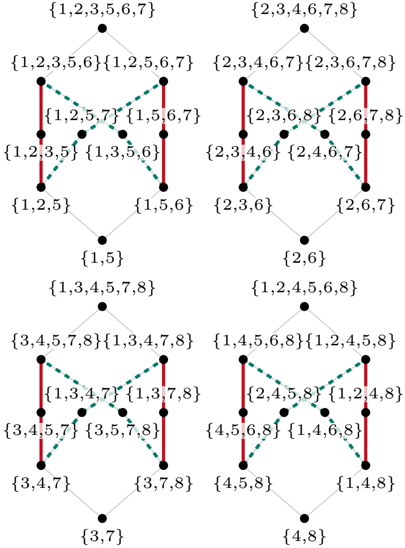


Edge-disjoint SCDs in N_8 that are not unrollable to edge-disjoint SCDs in Q_8





Small dimensions

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
almost-orthogonal SCDs	1	2	2	2	3	3*	4*	3*	3*	3	4*	3	3*	4*	3	3*	3	4*	3	3	4*	4*	3*	3	4*
edge-disjoint SCDs	1	2	2	3	3	4	4	4	4*	5*	6*	4	4	4	4	4	4	4	4	5*	5*	6*	4	4	4
upper bound $\lfloor n/2 \rfloor + 1$	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13	13